ON DERIVED TAME ALGEBRAS

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Abstract. Let Λ be a finite-dimensional algebra over an algebraically closed field k. We prove that $\mathcal{D}^b(\Lambda)$ the bounded derived category has tame representation type (Λ is called tame derived), if and only if the full subcategory of $\mathcal{D}^b(\Lambda)$ whose objects are perfect complexes is of tame representation type. We see that if Λ is derived tame then, almost all isomorphism classes of indecomposable complexes $X^\bullet \in \mathcal{D}^b(\Lambda)$ with fixed homology dimension are perfect and have Auslander-Reiten triangles of the form: $X^\bullet \to H^\bullet \to X^\bullet \to X^\bullet[1]$.

1. Introduction

Let Λ be a finite-dimensional algebra over an algebraically closed field k and $\mathcal{D}^b(\Lambda)$ be its bounded derived category. We consider $\operatorname{Mod}\Lambda$ the category of left Λ-modules. We denote by $\operatorname{mod}\Lambda$, $\operatorname{Proj}\Lambda$, $\operatorname{proj}\Lambda$, $\operatorname{Inj}\Lambda$ and $\operatorname{inj}\Lambda$ the full subcategories of $\operatorname{Mod}\Lambda$ consisting of the finitely generated, the projectives, the finitely generated projectives, the injectives and the finitely generated injectives Λ-modules, respectively. By $\mathcal{D}^b(\operatorname{Mod}\Lambda)$ we denote the bounded derived category of $\operatorname{Mod}\Lambda$, we recall that $\mathcal{D}^b(\Lambda)$ is the bounded derived category of the category $\operatorname{mod}\Lambda$. If $X = (X^i, d_X^i)_{i\in\mathbb{Z}}$ is an object in $\mathcal{D}^b(\Lambda)$ an invariant of it is given by its homology dimension $\mathbf{h}\dim = (h_i)_{i\in\mathbb{Z}}$ with $h_i = \dim_k H^i(X)$.

A sequence of non negative integers $\mathbf{h} = (h_i)_{i\in\mathbb{Z}}$ is called a homology dimension if for all but finitely many i, $h_i = 0$. We recall that according with [18], $\mathcal{D}^b(\Lambda)$ is called discrete and Λ derived discrete if there are only finitely many isoclasses of indecomposables $X \in \mathcal{D}^b(\Lambda)$ with fixed homology dimension. As for algebras, definitions of tame representation type and of wild representation type has been given in [12] for the category $\mathcal{D}^b(\Lambda)$. The algebra Λ is called derived tame or derived wild if the category $\mathcal{D}^b(\Lambda)$ is of tame representation type or of wild representation type, respectively.

In [18] it has been proved that Λ is derived discrete if and only if $\mathcal{D}^b(\Lambda)_{prf}$, the full subcategory of $\mathcal{D}^b(\Lambda)$ whose objects are the perfect complexes is discrete. We prove that a similar fact is also true for the tame case: Λ is derived tame if and only if $\mathcal{D}^b(\Lambda)_{prf}$ is of tame representation type. In fact we prove that almost all isomorphism classes of indecomposable objects in $\mathcal{D}^b(\Lambda)$ of given homology dimension are isomorphism classes of perfect objects.

Moreover we see that if Λ is derived tame and $\mathbf{h}$ is a fixed homology dimension, then for almost all isomorphism classes $[Y]$ with Y indecomposable perfect complex with $\mathbf{h}\dim Y = \mathbf{h}$, there is an Auslander-Reiten triangle of the form:

$$Y \to H \to Y \to Y[-1].$$

2000 Mathematics Subject Classification,16G70, 18G35, 18G05.

In addition, if $\mathbf{h} = (h_i)$, $Y = (Y^i, d_Y^i)$ and n_0 is the integer such that $h_{n_0} \neq 0$ and $h_i = 0$ for $i < n_0$, then $Y_j = 0$ for $j \leq n_0 - 1$ and $d_Y^{n_0-1} : Y^{n_0-1} \to Y^{n_0}$ is a monomorphism. This implies that for Λ derived tame for any fixed non-negative integer, almost all isomorphism classes of indecomposable Λ-modules $[M]$ with $\dim_k M \leq d$, the projective dimension of M is equal to one.

For the proof of the above results, we consider in section 2, $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ which is the category of complexes $X = (X^i, d_X^i)$ of finitely generated projective Λ-modules with $X^i = 0$ for i outside the interval $[1, ..., m]$. We denote by $\mathbf{C^1_m}(\mathrm{proj}\,\Lambda)$ the full subcategory of $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ whose objects are the complexes $X = (X^i, d_X^i)$ such that $\mathrm{Im} d_X^{i-1} \subset \mathrm{rad} X^i$ for all $i \in \mathbb{Z}$.

In general if $\mathcal{C}$ is a k-category a morphism $f : M \to N$ in $\mathcal{C}$ is called radical if for any split monomorphism $\sigma : X \to M$ and any split epimorphism $\pi : M \to Y$, $\pi f \sigma : X \to Y$ is not isomorphism. If P and Q are projective Λ-modules, $f : P \to Q$ is a radical morphism if and only if $\mathrm{Im} f \subset \mathrm{rad} Q$.

In section 6 we prove the following two results.

Theorem 1.1. *For fixed m, either $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ is of tame representation type or of wild representation type.*

The proof of this last result is in fact considered in [5] and [10], using bocses with relations. We present a different proof using just free triangular bocses. We recall from [2] that we have an exact category $(\mathbf{C_m}(\mathrm{proj}\,\Lambda), \mathcal{E})$ in the sense of [17] or [11], where $\mathcal{E}$ is the class of sequences of morphisms (conflations)

$$X \xrightarrow{u} E \xrightarrow{v} Y$$

such that for all $i \in \mathbb{Z}$ the sequence

$$0 \to X^i \xrightarrow{u^i} E^i \xrightarrow{v^i} Y^i \to 0,$$

is an split exact sequence. The exact category $(\mathbf{C_m}(\mathrm{proj}\,\Lambda), \mathcal{E})$ has enough projectives and injectives and it has almost split sequences.

Theorem 1.2. *Suppose $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ is of tame representation type. Then for almost all isomorphism classes $[X]$ of indecomposables with a fixed dimension $d = \dim_k X = \sum_i \dim_k X^i$ in the category $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$, there is an $\mathcal{E}$-almost split sequence in $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ of the form: $X \to E \to X$.*

For this we use in a similar way as in [5] tbocses (introduced in [1]).

In section 7 we consider generic complexes in $\mathcal{D}^b(\mathrm{Mod}\,\Lambda)$ in the sense of section 5 of [16], observe that this definition differs of the one given in [12]. With our definition we obtain similar results to the ones given in [8] for Λ-modules. In particular each generic complex is closely related to an one-parameter family of objects in $\mathcal{D}^b(\Lambda)$. In addition we prove that if X is a generic complex for a derived tame algebra Λ, X is isomorphic in $\mathcal{D}^b(\mathrm{Mod}\,\Lambda)$ to a bounded complex of projective Λ-modules.

2. Bounded derived categories

Here we see some consequences of Theorems 1.1 and 1.2 for the derived category $\mathcal{D}^b(\Lambda)$.

In the following a rational algebra is a k-algebra of the form:
$k[x]_h = \{f/h^m | m$ is a positive integer, $f \in k[x]\}$, the support of a rational algebra

is defined by $S(k[x]_h) = \{\lambda \in k | h(\lambda) \neq 0\}$. For $\lambda \in S(k[x]_h)$, the simple $k[x]_h$-module $k[x]/(x-\lambda)$ will be denoted by S_λ.

For $\mathbf{h}$ a homology dimension we denote by $\mathcal{V}(\mathbf{h})$ the full subcategory of $\mathcal{D}^b(\Lambda)$ whose objects are indecomposables $X \in \mathcal{D}^b(\Lambda)$ with $\mathbf{h}\text{dim}X = \mathbf{h}$.

We recall the following definitions:

1) Λ is called *derived discrete* if for each homology dimension $\mathbf{h}$, the category $\mathcal{V}(\mathbf{h})$ has only finitely many isomorphism classes.

2) Λ is called *derived tame* if for each homology dimension $\mathbf{h}$ there is a finite set of rational algebras $R_u, u = 1, ..., s$ and for each u a bounded complex M_u of $\Lambda - R_u$-bimodules free finitely generated over R_u, such that for almost all isomorphism classes $[X]$ with $X \in \mathcal{V}(\mathbf{h})$ there is a $\lambda \in S(R_u)$ with $X \cong M_u \otimes_{R_u} S_\lambda$ for some $u \in \{1, ..., s\}$.

3) Λ is called *derived wild* if there is a bounded complex W of $\Lambda - k < x, y >$-bimodules free finitely generated over $k < x, y >$ such that the functor

$$W \otimes_{k<x,y>} - : \text{mod}\, k < x, y > \to \mathcal{D}^b(\Lambda)$$

preserves isoclasses and indecomposables.

Concerning the categories $\mathbf{C_m}(\text{proj}\,\Lambda)$ we recall the definitions of finite representation type, tame representation type and wild representation type.

4) $\mathbf{C_m}(\text{proj}\,\Lambda)$ is called of *finite representation type* if it has only a finite number of isomorphism classes of indecomposables.

5) $\mathbf{C_m}(\text{proj}\,\Lambda)$ is called of *tame representation type* if for any given positive integer d there are rational algebras $R_u, u = 1, ..., s$ and for each u a complex $M_u = (M_u^i, d_{M_u}^i)$ with M_u^i a $\Lambda - R_u$-bimodule free finitely generated over R_u, projective as Λ-module and $M_u^i = 0$ for i outside the interval $[1, ..., m]$, such that for almost all isomorphism class $[Y]$ with Y indecomposable and $\text{dim}_k Y \leq d$ there is a $\lambda \in S(R_u)$ such that $M_u \otimes_{R_u} S_\lambda \cong Y$.

6) $\mathbf{C_m}(\text{proj}\,\Lambda)$ is called of *wild representation type* if there is a bounded complex of $\Lambda - k < x, y >$-bimodules free finitely generated over $k < x, y >$, projectives as Λ-modules, $W = (W^i, d_W^i)$ with $W^i = 0$ for i outside the interval $[1, ..., m]$, such that the functor:

$$W \otimes_{R_u} - : \text{mod}\, k < x, y > \to \mathbf{C_m}(\text{proj}\,\Lambda)$$

preserves isoclasses and indecomposables.

We need the following results.

Lemma 2.1. *Suppose* $Y = (Y^i, d_Y^i) \in \mathbf{C^1_m}(\text{proj}\,\Lambda)$ *is such that* $\text{dim}_k H^j(Y^\bullet) \leq c$ *for all* j *and for some* $u \in [2, ..., m]$, $\text{dim}_k Y^u \leq d_u$, *then* $\text{dim}_k Y^{u-1} \leq (d_u + c)L$, *with* $L = \text{dim}_k \Lambda$.

Proof. We have $\text{dim}_k Y^{u-1}/\text{Ker}d_Y^{u-1} = \text{dim}_k \text{Im}d_Y^{u-1} \leq d_u$, moreover we know that $\text{dim}_k \text{Ker}d_Y^{u-1}/\text{Im}d_Y^{u-2} \leq c$. Therefore $\text{dim}_k Y^{u-1}/\text{Im}d_Y^{u-2} \leq c + d_u$.

Here $\text{Im}d_Y^{u-2} \subset \text{rad}Y^{u-1}$, thus $\text{dim}_k Y^{u-1}/\text{rad}Y^{u-1} \leq \text{dim}_k Y^{u-1}/\text{Im}d_Y^{u-2}$. Consequently, $\text{dim}_k Y^{u-1} \leq (c + d_u)L$. □

Lemma 2.2. *Let* $Y^\bullet = (Y^i, d_Y^i) \in \mathbf{C^1_m}(\text{proj}\,\Lambda)$ *such that for all* j, *we have the inequality* $\text{dim}_k H^j(Y^\bullet) \leq c$ *for some fixed* c. *Then*

$$\text{dim}_k Y \leq c(mL + (m-1)L^2 + (m-2)L^3 + ... + 2L^{m-1} + L^m).$$

Proof. Here $Y^{m+1} = 0$, then by our previous lemma, $\dim_k Y^m \leq cL$. Then again by lemma 2.1 we have, $\dim_k Y^{m-1} \leq c(L+L^2)$, $\dim_k Y^{m-2} \leq c(L+L^2+L^3)$, ..., $\dim_k Y^1 \leq c(L+L^2+...+L^m)$. From here we obtain our result. □

We denote by $\mathbf{C}^{\leq \mathbf{m},\mathbf{b}}(\mathrm{Proj}\,\Lambda)$ the category of complexes $X = (X^i, d_X^i)$ with $X^i \in \mathrm{Proj}\,\Lambda$ and $X^i = 0$ for $i > m$, such that $H^i(X) = 0$ for almost all i. By $\mathbf{K}^{\leq \mathbf{m},\mathbf{b}}(\mathrm{Proj}\,\Lambda)$ we denote the corresponding homotopy category.

Following [2] we denote by $\mathcal{L}_m$ the full subcategory of $\mathbf{K}^{\leq \mathbf{m},\mathbf{b}}(\mathrm{Proj}\,\Lambda)$ whose object are those X with $H^i(X) = 0$ for $i \leq 1$.

The functor $F : \mathbf{K}^{\leq \mathbf{m},\mathbf{b}}(\mathrm{Proj}\,\Lambda) \rightarrow \mathrm{C_m}(\mathrm{Proj}\,\Lambda)$ which sends a complex:

$$X : ... \rightarrow X^{-1} \stackrel{d^{-1}}{\rightarrow} X^0 \stackrel{d^0}{\rightarrow} X^1 \stackrel{d^1}{\rightarrow} ... \rightarrow X^m \rightarrow 0$$

to

$$F(X) = ...0 \rightarrow 0 \rightarrow X^1 \stackrel{d^1}{\rightarrow} ... \rightarrow X^m \rightarrow 0,$$

induces an equivalence:

$$\underline{F} : \mathcal{L}_m \rightarrow \overline{\mathbf{C_m}}(\mathrm{Proj}\,\Lambda),$$

where $\overline{\mathbf{C_m}}(\mathrm{Proj}\,\Lambda)$ is the category with the same objects as $\mathbf{C_m}(\mathrm{Proj}\,\Lambda)$ and morphisms those in $\mathbf{C_m}(\mathrm{Proj}\,\Lambda)$ modulo the ones which are factorized through $\mathcal{E}$-injective objects (see Corollary 5.7 of [2]).

Moreover we have an embedding

$$\tau^{\geq 1} : \mathcal{L}_m \rightarrow \mathcal{D}^b(\mathrm{Mod}\,\Lambda).$$

Observe that for $P \in \mathcal{L}_m$, $q : P \rightarrow \tau^{\geq 1}P$ the natural morphism is a quasi-isomorphism.

For a natural number d we denote by $\mathcal{F}_d$ the full subcategory of $\overline{\mathbf{C_m}}(\mathrm{proj}\,\Lambda)$ whose objects are those indecomposables X with $\dim_k X \leq d$. We denote by $\mathcal{U}_d$ the full subcategory of $\mathcal{L}_m$ whose objects are those $Y \cong F(P)$ with $P \in \mathcal{F}_d$. By $\mathcal{V}_d$ we denote the full subcategory of $\mathcal{D}^b(\Lambda)$ whose objects are those isomorphic to some $\tau^{\geq 1}P$ with $P \in \mathcal{U}_d$.

We have $\mathcal{V}(\mathbf{h}) \subset \mathcal{V}_d$, if $d = |\mathbf{h}|(mL + (m-1)L^2 + ... + 2L^{m-1} + L^m)$ with $|\mathbf{h}| = max\{h_i\}_{i \in \mathbb{Z}}$, $L = \dim_k \Lambda$.

Theorem 2.3. *a) Λ is derived discrete if and only if for all m, $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ is of finite representation type;*
b) if Λ is derived wild it is not derived tame;
c) if for some m, $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ is of wild representation type then Λ is derived wild;
d) Λ is derived tame if and only if for all m, $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$, is of tame representation type;
e) Λ is either derived tame or derived wild (see Bekkert-Drozd [5]).

Proof. Suppose Λ is derived discrete, then by [18] Λ is derived hereditary of Dynkin type or it is a gentle algebra.

For a Krull-Schmidt category $\mathcal{C}$ we denote by $ind\,\mathcal{C}$ the full subcategory of $\mathcal{C}$ whose objects are the indecomposables of $\mathcal{C}$.

If Λ is hereditary then $\mathbf{C_2}(\mathrm{proj}\,\Lambda)$ is of finite representation type, for $m > 2$ we have:

$$ind\,\mathbf{C_m}(\mathrm{proj}\,\Lambda) \subset ind\,\mathbf{C_2}(\mathrm{proj}\,\Lambda) \cup ind\,\mathbf{C_2}(\mathrm{proj}\,\Lambda)[1] \cup ... \cup ind\,\mathbf{C_2}(\mathrm{proj}\,\Lambda)[m-1]$$

then $ind\,\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ has only finitely many isomorphism classes, thus it is of finite representation type.

If Λ is derived equivalent to a hereditary algebra A of Dynkin type, there is a bounded complex T over $\Lambda - A$-bimodules projective finitely generated over both sides such that the functor:

$$- \otimes^{\mathbf{L}} T : \mathcal{D}^b(\Lambda) \to \mathcal{D}^b(A)$$

is an equivalence. Then for m there is a n and a l such that we have a functor:

$$G(-) = - \otimes_\Lambda T[l] : \mathbf{C_m}(\operatorname{proj} \Lambda) \to \mathbf{C_{m+n}}(\operatorname{proj} A)$$

with the following property: if Y and X are indecomposables in $\mathbf{C_m}(\operatorname{proj}\Lambda)$ which are not $\mathcal{E}$-injectives or $\mathcal{E}$-projectives then their images under G are also indecomposables and $G(Y) \cong G(X)$ imply $Y \cong X$. Here $\mathbf{C_{m+n}}(\operatorname{proj} A)$ is of finite representation type, then also $\mathbf{C_m}(\operatorname{proj}\Lambda)$ is of finite representation type.

Now suppose that Λ is a gentle algebra $k(Q, I)$. Then from the description of the objects in $\mathbf{K}^{-,\mathbf{b}}(\operatorname{proj}\Lambda)$ in [6] one can see that if there are generalized strings in Q of arbitrary size corresponding to complexes in $\mathbf{C_m}(\operatorname{proj}\Lambda)$ for some fixed m, then there are generalized bands, but this implies that Λ is not derived discrete, therefore for any m, $\mathbf{C_m}(\operatorname{proj}\Lambda)$ is of finite representation type.

Conversely assume $\mathbf{C_m}(\operatorname{proj}\Lambda)$ is of finite representation type for all m.

Take $\mathbf{h} = (h_i)$ a homology dimension, we may assume $h_i = 0$ for i outside the interval $[2, .., m]$. Take $d = |\mathbf{h}|(mL+(m-1)L^2+...+2L^{m-1}+L^m)$, then by Lemma 2.2, $\mathcal{V}(\mathbf{h}) \subset \mathcal{V}_d$. The categories $\mathcal{V}_d$, $\mathcal{U}_d$ and $\mathcal{F}_d$ are equivalent, by assumption $\mathcal{F}_d$ has only a finite number of isoclasses, the same is true for $\mathcal{V}(\mathbf{h})$. Therefore Λ is derived discrete.

The part b) is proved in Theorem 5.2 of [12].

c) Suppose that $\mathbf{C_m}(\operatorname{proj}\Lambda)$ is of wild representation type. Then there is a bounded complex $W = (W^i, d^i_W)$ of $\Lambda - k < x, y >$-bimodules free finiteley generated over the right side, projectives as Λ-modules, with $W^i = 0$ for i outside the interval $[1, ..., m]$ and $\operatorname{Im} d^{i-1}_W \subset \operatorname{rad}\Lambda W^i$, such that the functor $W \otimes_{k<x,y>} - :$ $\operatorname{mod} k < x, y > \to \mathbf{C_m}(\operatorname{proj}\Lambda)$ preserves iso-classes and indecomposables. The composition of this functor with the composition $\mathbf{C_m}(\operatorname{proj}\Lambda) \to \mathbf{K}^{-,\mathbf{b}}(\operatorname{proj}\Lambda) \to \mathcal{D}^b(\Lambda)$ also preserves iso-classes and indecomposables, consequently Λ is derived wild.

d) Suppose Λ is derived tame, then if for some m, $\mathbf{C_m}(\operatorname{proj}\Lambda)$ is of wild representation type then by c), Λ is derived wild, which contradicts b). Therefore for all m, $\mathbf{C_m}(\operatorname{proj}\Lambda)$ is not of wild representation type, but this implies, by Theorem 1.1 that for all m, $\mathbf{C_m}(\operatorname{proj}\Lambda)$ is of tame representation type.

Conversely assume that for all m, $\mathbf{C_m}(\operatorname{proj}\Lambda)$ is of tame representation type. Let $\mathbf{h}$ be a fixed homology dimension, take $d = |\mathbf{h}|(mL+(m-1)L^2+...+2L^{m-1}+L^m)$ then $\mathcal{V}(\mathbf{h}) \subset \mathcal{V}_d$. Therefore there are rational algebras R_u, $u = 1, ..., s$ and for each u a bounded complex $M_u = (M^i_u, d^i_{M_u})$ over the $\Lambda - R_u$-bimodules free finitely generated over the right side with $M^i_u = 0$ for i outside the interval $[1, ..., m]$ such that for almost all isomorphism class $[X]$ in $\mathcal{F}_d$ there is a u and $\lambda \in S(R_u)$ with $X \cong M_u \otimes_{R_u} S_\lambda$.

We may assume that for all u and i, $\operatorname{Im} d^{i-1}_{M_u}$ and $\operatorname{Ker} d^i_{M_u}$ are direct summands of M^i_u as right R_u-modules.

Then for each u, $W_u = \tau^{\geq 1} M_u$ is a bounded complex over the $\Lambda - R_u$-bimodules which is free finitely generated over the right side.

Take $Y \in \mathcal{V}(\mathbf{h})$, then there is a $P \in \mathcal{U}_d$ with a quasi-isomorphism $q : P \to Y$, we have $\tau^{\geq 1} P \cong Y$ in $\mathcal{D}^b(\Lambda)$.

Clearly $\tau^{\geq 1}P = \tau^{\geq 1}F(P)$, $F(P) \in \mathcal{F}_d$. Therefore $F(P) \cong M_u \otimes_{R_u} S_\lambda$ for some u and some $\lambda \in S(R_u)$. Thus

$$Y \cong \tau^{\geq 1}P = \tau^{\geq 1}F(P) \cong \tau^{\geq 1}(M_u \otimes_{R_u} S_\lambda) \cong \tau^{\geq 1}(M_u) \otimes_{R_u} S_\lambda = W_u \otimes_{R_u} S_\lambda.$$

consequently Λ is derived tame.

e) Suppose Λ is not derived wild, then by c) for all m, $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ is not of wild representation type, by Theorem 1.1, for all m, $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ is of tame representation type. Therefore by d), Λ is derived tame. □

Theorem 2.4. *Let Λ be a derived tame algebra and $\boldsymbol{h} = (h_i)$ be a fixed homology dimension such that for n_0, $h_{n_0} \neq 0$ and $h_i = 0$ for $i < n_0$. Then for almost all isomorphism class of indecomposable objects $X \in \mathcal{D}^b(\Lambda)$ with $\boldsymbol{h}\mathrm{dim}X = \boldsymbol{h}$, X is a perfect object and there is an Auslander-Reiten triangle of the form:*

$$X \to H \to X \to X[1].$$

Moreover if $X = (X^i, d_X^i)$ then $X_i = 0$ for $i < n_0 - 1$ and $d_X^{n_0-1} : X^{n_0-1} \to X^{n_0}$ is a monomorphism.

Proof. After a shifting we may assume $h_i = 0$ for $i \leq 1$ and $i > n$, $h_2 \neq 0$. By $\mathcal{U}(\mathbf{h})$ we denote the full subcategory of $\mathbf{K}^{\leq \mathbf{n},\mathbf{b}}(\mathrm{proj}\,\Lambda)$ whose objects are quasi-isomorphic to complexes $X \in \mathcal{V}(\mathbf{h})$. The categories $\mathcal{U}(\mathbf{h})$ and $\mathcal{V}(\mathbf{h})$ are equivalent. We will see that for almost all isomorphism classes of objects P in $\mathcal{U}(\mathbf{h})$, P is a finite complex. If $P \in \mathcal{U}(\mathbf{h})$ then $\mathbf{h}\mathrm{dim}P = \mathbf{h}$, thus $\dim_k H^1(P) = h_1 = 0$, therefore $\mathcal{U}(\mathbf{h}) \subset \mathcal{L}_n$.

Recall that we have an equivalence $\underline{F} : \mathcal{L}_n \to \overline{\mathbf{C_n}}(\mathrm{proj}\,\Lambda)$.

Denote by $\mathcal{F}(\mathbf{h})$ the full subcategory of $\overline{\mathbf{C_n}}(\mathrm{proj}\,\Lambda)$ whose objects are isomorphic to some $\underline{F}(P)$ with $P \in \mathcal{U}(\mathbf{h})$. The categories $\mathcal{U}(\mathbf{h})$ and $\mathcal{F}(\mathbf{h})$ are equivalent categories. By Lemma 2.2, $\mathcal{F}(\mathbf{h}) \subset \mathcal{F}_d$ for $d = |\mathbf{h}|(nL + (n-1)L^2 + ...2L^{n-1} + L^n)$.

For our purposes it is convenient consider $\mathcal{F}(\mathbf{h})[-1]$ as a full subcategory of $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ with $m = n+3$. If $Y = (Y^i, d_Y^i)_{i\in\mathbb{Z}} \in \mathcal{F}(\mathbf{h})[-1]$, then $Y^1 = 0, Y^{n+2} = 0, Y^{n+3} = 0$ and $\dim_k Y \leq d$.

By d) of Theorem 2.1 $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ is of tame representation type. Then by Theorem 1.2 for almost all isomorphism class $[Y]$ with $Y \in \mathbf{C_m}(\mathrm{proj}\,\Lambda)$ there is an almost split conflation

$$Y \to E \to Y$$

in $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$.

Following the notation of [2] we recall that $A(Y) \cong Y$. In order to calculate $A(Y)$ we take $Z = (Z^i, d_Z^i)_{i\in\mathbb{Z}} = \nu(Y)[-1]$ and a quasi-isomorphism $q : Q = (Q^i, d_Q^i)_{i\in\mathbb{Z}} \to \tau^{\leq m}Z$, with $Q \in \mathbf{C}_{\mathbf{n}}^{\leq \mathbf{m},\mathbf{b}}(\mathrm{proj}\,\Lambda)$. Then $A(Y) \cong F(Q)$. Moreover by [14] there is an Auslander-Reiten triangle in $\mathcal{D}^b(\Lambda)$:

$$Z \to G \to Y \to Z[1].$$

We have $Z^m = Z^{n+3} = \nu(Y^{n+2}) = 0$, therefore $\tau^{\leq m}Z = Z$.

Here Z is indecomposable, then Q is an indecomposable complex in the category $\mathbf{K}^{\leq \mathbf{m},\mathbf{b}}(\mathrm{proj}\,\Lambda)$, we may choose Q an indecomposable object in the category $\mathbf{C}^{\leq \mathbf{m},\mathbf{b}}(\mathrm{proj}\,\Lambda)$ with $Q^m = 0$, here $Z^m = 0$.

We have $F(Q) \cong A(Y) \cong Y$ in $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$, thus, $Q^1 \cong Y^1 = 0$. Here Q is indecomposable, this implies that $Q^i = 0$ for $i \leq 1$. Moreover $Z^2 = \nu(Y^1) = 0$, then

$H^2(Q) \cong H^2(Z) = 0$. Therefore the morphism $d_Q^2 : Q^2 \to Q^3$ is a monomorphism and $Q \cong Y$, and $Z \cong Q \cong Y$ in $\mathcal{D}^b(\Lambda)$.

Thus we have an Auslander-Reiten triangle in $\mathcal{D}^b(\Lambda)$:

$$(*) \quad Y \to G \to Y \to Y[1].$$

Now $Y[1] \in \mathcal{F}(\mathbf{h})$ then $Y[1] \cong F(P)$ with $P \in \mathcal{U}(\mathbf{h})$. Therefore $P^1 \cong Y^2 \cong Q^2, P^2 \cong Y^3 \cong Q^3$. The morphism $d_Q^2 : Q^2 \to Q^3$ is isomorphic to the morphism $d_P^1 : P^1 \to P^2$, thus this last morphism is a monomorphism.

Here $h_1 = \dim_k(\mathrm{Ker} d_P^1/\mathrm{Im} d_P^0) = 0$, then $\mathrm{Im} d_P^0 = \mathrm{Ker} d_P^1 = 0$, consequently $d_P^0 = 0$. But P is indecomposable, therefore $P^i = 0$ for $i \leq 0$. Consequently $P = F(P) \cong Y[-1]$. Thus applying the equivalence $[-1]$ to $(*)$ we obtain our result. □

Corollary 2.5. *Suppose Λ is selfinjective, then either it is derived discrete or derived wild.*

Proof. Suppose Λ is neither derived discrete nor derived wild. Then there are infinitely many isomorphism classes in $\mathcal{V}(\mathbf{h})$ for some homology dimension $\mathbf{h}$. Therefore there is an indecomposable X in $\mathcal{D}^b(\Lambda)$ with an Auslander-triangle of the form $X \to H \to X \to X[1]$ with $X = (X^i, d_X^i)$ indecomposable object in $\mathbf{C^1_m}(\mathrm{proj}\,\Lambda)$ and $d_X^1 : X^1 \to X^2$ is a monomorphism, since X^1 is injective, this is not possible. □

Corollary 2.6. *Let Λ be derived tame, then for a fixed homology dimension $\boldsymbol{h}$, for almost all isomorphism classes $[X]$ with $X \in \mathcal{D}^b(\Lambda)$ a finite complex of finitely generated projectives and $\boldsymbol{h}\mathrm{dim}_k X = \boldsymbol{h}$, X is isomorphic to a finite complex of finitely generated injectives.*

Remark. Observe that gentle algebras are Gorenstein and in this case all finite complexes of finitely generated projectives are also isomorphic to finite complexes of finitely generated injectives (see [13]).

Corollary 2.7. *Let Λ be a derived tame algebra. Suppose P is a bounded complex of $\Lambda - R$-bimodules projectives over Λ and free finitely generated over R, a rational algebra, such that for all $\lambda \in S(R)$, $P \otimes_R S_\lambda$ is indecomposable in $\mathcal{D}^b(\Lambda)$, and for $\lambda \neq \mu \in S(R)$, $P \otimes_R S_\lambda \ncong P \otimes_R S_\mu$ in $\mathcal{D}^b(\Lambda)$. Then if $\boldsymbol{h}\mathrm{dim}_{k(x)} P \otimes_R k(x) = \boldsymbol{h} = (h_i)$ is such that $h_{n_0} \neq 0$ and $h_j = 0$ for $j < n_0$, we obtain that the morphism $d_P^{n_0-1} \otimes 1 : P^{n_0-1} \otimes_R k(x) \to P^{n_0} \otimes_R k(x)$ is a monomorphism .*

Proof We may assume that for all $\lambda \in S(R)$, all $\mathrm{Ker} d^i$ are direct summands of P^i as right R-modules. Thus $\mathbf{h}\mathrm{dim} P \otimes_R S_\lambda = \mathbf{h}$ for all $\lambda \in S(R)$. By Theorem 2.2, we may also assume that for all $\lambda \in S(R)$, $P^i \otimes S_\lambda = 0$ for $i < n_0 - 1$ and $\mathrm{Ker}(d^{n_0-1} \otimes 1 : P^{n_0-1} \otimes S_\lambda \to P^{n_0} \otimes S_\lambda) = 0$. But this implies our assertion. □

Corollary 2.8. *Suppose Λ is a derived tame algebra and d a fixed non-negative integer, then almost all isomorphism classes of indecomposable Λ-modules M with $\mathrm{dim}_k M = d$ have projective dimension one.*

Proof. For M indecomposable with $\dim_k M = d$, take

$$... \to P_M^{-3} \stackrel{d_M^{-3}}{\to} P_M^{-2} \stackrel{d_M^{-2}}{\to} P_M^{-1} \stackrel{d_M^{-1}}{\to} P_M^0 \stackrel{\eta}{\to} M \to 0$$

a minimal projective resolution of M. Consider $P_M = (P_M^j, d_M^j)$ with $P_M^j = 0$, for $j > 0$ and $d_M^j = 0$ for $j \geq 0$. Then for $\mathbf{h}\dim M = (h_i)$, we have $h_0 = d$, $h_j = 0$ for $j < 0$. Then by Theorem 2.4 for almost all isomorphism classes $[M]$, $P_M^j = 0$ for $j < -1$. This proves our claim. □

3. Bocses

A tbocs is a triple $\mathcal{A} = (R, W, \delta)$, where R is a k-algebra (k is a field), W is a R-bimodule such that $W = W_0 \oplus W_1$ as R bimodules. The elements of W_i are called homogeneous of degree i, $i \in \{0, 1\}$. For $w \in W_i$, we put $deg(w) = i$.

Take now $T_R(W)$ the tensor algebra:

$$R \oplus W \oplus W^{\otimes^2} \oplus ...$$

with the graduation induced by the one of W. The R-module generated by the set of homogeneous elements in $T_R(W)$ of degree i will be denoted by $T_R(W)_i$. Then δ is a endomorphism of R-bimodules of $T_R(W)$ such that

i) $\delta(T_R(W)_i) \subset T_R(W)_{i+1}$

ii) For a, b homogeneous elements of $T_R(W)$

$$\delta(ab) = \delta(a)b + (-1)^{dega} a\delta(b) \qquad \text{(Leibnitz rule)}$$

iii) $\delta^2 = 0$

The set of all elements of degree zero, $T_R(W)_0$ is a k-algebra which will be denoted by $A(\mathcal{A})$. This algebra is identified with $T_R(W_0)$. The set of all elements of degree one $T_R(W)_1$ is an $A(\mathcal{A})$-bimodule, which can be identified with $A(\mathcal{A}) \otimes_R W_1 \otimes_R A(\mathcal{A})$, and will be denoted by $V(\mathcal{A})$. Thus $T_R(W)$ is a differential graded algebra with differential δ. For v_1, v_2 in $T_R(W)$ we denote its product by v_1v_2, in particular if the above elements are in W, $v_1v_2 = v_1 \otimes v_2$.

Let $\mathcal{A} = (R, W, \delta)$ be a tbocs. The category of representations of $\mathcal{A}$, Rep$\mathcal{A}$ is defined as follows:

The objects of Rep($\mathcal{A}$) are the left $A(\mathcal{A})$-modules.

Given two $A(\mathcal{A})$-modules M and N, a morphism $f : M \to N$ in Rep$\mathcal{A}$ is given by a pair $f = (f^0, f^1)$, where

$$f^0 \in \mathrm{Hom}_R(M, N), \quad f^1 \in \mathrm{Hom}_{A(\mathcal{A}),A(\mathcal{A})}(V(\mathcal{A}), \mathrm{Hom}_k(M, N))$$

such that for all $a \in A(\mathcal{A}), m \in M$:

$$af^0(m) = f^0(am) + f^1(\delta(a))(m).$$

Observe that the pair $(f^0, 0)$ is a morphism in Rep$\mathcal{A}$ iff f^0 is a $A(\mathcal{A})$-morphism. Now if $f = (f^0, f^1) : M \to N$ and $g = (g^0, g^1) : N \to L$ are morphisms in Rep$\mathcal{A}$, the pair given by $(g^0f^0, (gf)^1))$ with

$$(gf)^1(v) = g^1(v)f^0 + g^0f^1(v) + \sum_{i=1}^{l} g^1(v_i^1)f^1(v_i^2)$$

for $\delta(v) = \sum_{i=1}^{l} v_i^1v_i^2$, $v_i^1, v_i^2 \in V(\mathcal{A})$, is again a morphism. We will put $gf = (g^0f^0, (gf)^1)$.

Using the properties of δ one can see that Rep$\mathcal{A}$ is a category. The identity morphism for $M \in$ Rep$\mathcal{A}$ is given by the pair $\underline{id_M} = (id_M, 0)$.

For a tbocs $\mathcal{A} = (R, W, \delta)$ we have a functor

$$I_{\mathcal{A}} : \mathrm{Mod}\, A(\mathcal{A}) \to \mathrm{Rep}\,\mathcal{A}$$

which is the identity on objects and for morphisms $u : M \to N$ of $A(\mathcal{A})$-modules, we have $I_{\mathcal{A}}(u) = (u, 0)$.

Let S be a k-algebra containing S_0 as k-subalgebra. We assume S_0 is a basic semisimple finite dimensional k-algebra, $1 = \sum_{i=1}^{n} e_i$ a decomposition into central orthogonal primitive idempotents.

Definition 3.1. *Let W be a S-bimodule. A S_0-subimodule $\tilde{W}$ of W is said to be a S_0-free generator of W if any morphism of S_0-bimodules $u : \tilde{W} \to V$, V a S-bimodule has a unique extension to a morphism of S-bimodules $v : W \to V$. In this case we say that W is a S_0-free S-bimodule.*

It is easy to see that $\tilde{W}$ is a S_0-free generator of W iff the morphism

$$\rho : S \otimes_{S_0} \tilde{W} \otimes_{S_0} S \to W \text{ given by } \rho(s \otimes w \otimes s_1) = sws_1$$

is an isomorphism. On the other hand if $\sigma : S \otimes_{S_0} \tilde{W} \otimes_{S_0} S \to W$ is an isomorphism $\sigma(\tilde{W})$ is a S_0-free generator of W.

Definition 3.2. *A tbocs $\mathcal{A} = (S, W, \delta)$ is called S_0-free triangular if the following conditions are satisfied:*
T.1 There is a filtration of S-bimodules $\{0\} = W_0^0 \subset ... \subset W_0^r = W_0$ such that for $i \geq 1$ $\delta(W_0^i) \subset A_i W_1 A_i$, where A_i is the R-subalgebra of A generated by W_0^{i-1}.
T.2 There is a filtration of S_0-bimodules $\tilde{W}_0^1 \subset ... \subset \tilde{W}_0^r = \tilde{W}_0$ such that $\tilde{W}_0^j$ is a S_0-free generator of W_0^j.
T.3 There is a sequence of subbimodules $\{0\} = W_1^0 \subset ... \subset W_1^s = W_1$ such that for $i \geq 1$ $\delta(W_1^i) \subset AW_1^{i-1}AW_1^{i-1}A$.
T.4 W_1 is S_0-freely generated by $\tilde{W}_1$.
If a tbocs $\mathcal{A}$ satisfies T.1, T.2 and T.4, we say that $\mathcal{A}$ is weakly triangular.

Through the paper S_0-free triangular tbocses will be called simply triangular tbocses. We recall that in the category Rep$\mathcal{A}$ idempotents split, moreover for $f = (f^0, f^1) : M \to N$, f is an isomorphism if and only if f^0 is an isomorphism.

Definition 3.3. *The k-algebra S is called minimal if there is a decomposition $1 = \sum_i e_i$ into central orthogonal primitive idempotents, such that $e_i S = e_i k$ or $e_i S$ is a rational k-algebra.*

Definition 3.4. *The tbocs $\mathcal{A} = (R, W, \delta)$ is called minimal if R is a minimal k-algebra and $W_0 = 0$.*

If $\mathcal{A} = (R, W, \delta)$ is a minimal tbocs then $A(\mathcal{A}) = R, V(\mathcal{A}) = W$, for $M, N \in$ Rep$\mathcal{A}$ the morphisms from M to N are given by all pairs $f = (f^0, f^1)$ with $f^0 \in \mathrm{Hom}_R(M, N)$, $f^1 \in \mathrm{Hom}_{R-R}(W, \mathrm{Hom}_k(M, N))$.

Lemma 3.5. *Suppose $\mathcal{A} = (R, W, \delta)$ is a triangular minimal tbocs, and $f : M \to M$ a morphism in* Rep$\,\mathcal{A}$ *of the form $f = (0, f^1)$, then f is nilpotent.*

Proof. Take $0 = W^0 \subset W^1 \subset ... \subset W^s = W$, the filtration of $W = W_1$ given by condition T.3 of Definition 3.2. Then we have $f^2 = (0, (f^2)^1)$ and $(f^2)^1(W^1) = 0$.

In general $f^r = (0, (f^r)^1)$ and $(f^r)^1(W^{r-1}) = 0$, therefore $f^{s+1} = (0, (f^{s+1})^1)$ and $(f^{s+1})^1(W^s) = (f^{s+1})^1(W) = 0$. Consequently $f^{s+1} = 0$. □

Proposition 3.6. *Suppose $\mathcal{A} = (R, W, \delta)$ is a triangular minimal tbocs, then an object $M \in \mathrm{Rep}\mathcal{A}$ is indecomposable if and only if ${}_RM$ is indecomposable.*

Proof. If M is indecomposable in $\mathrm{Rep}\mathcal{A}$, clearly ${}_RM$ is indecomposable. Suppose now that ${}_RM$ is indecomposable. Take $f = (f^0, f^1)$ an idempotent element in $\mathrm{End}_{\mathcal{A}}(M)$. Then $(f^0)^2 = f^0$, thus $f^0 = 0$ or $f^0 = id_M$. In the first case $f = (0, f^1)$, thus f is nilpotent, then since f is also idempotent we conclude that $f = 0$. In the second case f is an isomorphism therefore there is a $g \in \mathrm{End}_{\mathcal{A}}(M)$ with $fg = gf = id_M$. Then $id_M = fg = f^2g = f(fg) = f$. Therefore M is indecomposable in $\mathrm{Rep}\mathcal{A}$. This proves our result. □

For $\mathcal{A} = (R, W, \delta)$ a minimal tbocs, take $1_R = \sum_{i=1}^n e_i$ a decomposition of 1_R as a sum of central primitive orthogonal idempotents.

Proposition 3.7. *Suppose $\mathcal{A} = (R, W, \delta)$ is a minimal triangular tbocs. Then if $M \in \mathrm{Rep}\mathcal{A}$ is indecomposable there is an e_i with $e_iM = M$*

Proof. Here $R \cong Re_1 \times ... \times Re_n$, if M is an indecomposale R-module then $e_iM = M$ for some e_i. Our result follows from our previous proposition. □

4. Reduction Functors

In this section we study full and faithful functors $F : \mathrm{Rep}\mathcal{B} \to \mathrm{Rep}\mathcal{A}$ which have been considered in [1].

Let R be a k-algebra, we recall from [1] that X a left R-module is called $R - R_X$ admissible if R_X is a k-subalgebra of $\mathrm{End}_R(X)^{op}$ such that $\mathrm{End}_R(X)^{op} = R_X \oplus \mathcal{R}$ as R_X-bimodules with $\mathcal{R}$ an ideal of $\mathrm{End}_R(X)^{op}$, finitely generated projective as right R_X-module, and X finitely generated projective as right R_X-module. We have $X^* = \mathrm{Hom}_{R_X}(X_{R_X}, R_X)$ is a $R_X - R$-bimodule and $\mathcal{R}^* = \mathrm{Hom}_{R_X}(\mathcal{R}_{R_X}, R_X)$ is a R_X-bimodule. Take dual bases $\{p_j, \gamma_j\}$ for $\mathcal{R}$ and $\{x_i, u_i\}$ for X as right R_X-modules.

We have morphisms

$$e : X \to X \otimes_{R_X} \mathcal{R}^*, \quad a : X^* \to \mathcal{R}^* \otimes_{R_X} X^*$$

such that for $u \in X^*, x \in X$, we have

$$e(x) = -\sum_j p_j(x) \otimes \gamma_j, \quad a(u) = \sum_{i,j} u(p_j(x_i))\gamma_j \otimes u_i.$$

Let $\mathcal{A} = (R, W, \delta)$ be a tbocs and X a $R-R_X$ admissible left R-module. Consider the R_X-bimodules $(W_X)_0 = X^* \otimes_{R_X} W_0 \otimes_{R_X} X$, $(W_X)_1 = (X^* \otimes_{R_X} W_1 \otimes_{R_X} X) \oplus \mathcal{R}^*$.

For $u \in X^*$ and $v \in X$ we have k-linear maps:

$$\phi^0_{u,v} : R \to R_X,$$

for $n \geq 1$:

$$\phi^n_{u,v} : W^{\otimes^n} \to T_{R_X}(W_X)$$

given by $\phi^0_{u,v}(r) = u(rv)$, $\phi^n_{u,v}(w_1 \otimes w_2 \otimes ... \otimes w_n) =$
$\sum_{i_1,i_2,...,i_{n-1}} u \otimes w_1 \otimes x_{i_1} \otimes u_{i_1} \otimes w_2 \otimes x_{i_2} \otimes u_{i_2} \otimes ... \otimes x_{i_{n-1}} \otimes u_{i_{n-1}} \otimes w_n \otimes v.$

These morphisms determine a k-linear map:

$$\phi_{u,v} : T_R(W) \to T_{R_X}(W_X),$$

such that for $\lambda_1, \lambda_2 \in T_R(W)$ we have $\phi_{u,v}(\lambda_1\lambda_2) = \sum_i \phi_{u,x_i}(\lambda_1)\phi_{u_i,v}(\lambda_2)$. For $u \in X^*, v \in X$ we put for $\lambda \in T_R(W)$, $\phi_{a(u),v}(\lambda) = \sum_{i,j} u(p_j(x_i))\gamma_j\phi_{u_i,v}(\lambda)$ and $\phi_{u,e(v)}(\lambda) = -\sum_j \phi_{u,p_j(x)}(\lambda)\gamma_j$.

There is a differential δ_X in $T_{R_X}(W_X)$ with $\delta_X^2 = 0$, and such that for t a homogeneous element in $T_R(W)^1 = W \oplus W^{\otimes^2} \oplus ...$ and $u \in X^*$, $v \in X$

$$(*) \quad \delta_X(\phi_{u,v}(t)) = \phi_{a(u),v}(t) + \phi_{u,v}(\delta(t)) + (-1)^{degt}\phi_{u,e(v)}(t).$$

For $r \in R, u \in X^*, v \in X$, we have:

$$\phi_{a(u),v}(r) + \phi_{u,e(v)}(r) = \sum_{i,j} u(p_j(x_i))\gamma_j u_i(rv) - \sum_j u(rp_j(v))\gamma_j$$

$$= \sum_{i,j} u(p_j(x_i u_i(rv)\gamma_j - \sum_j u(p_j(rv)\gamma_j = 0.$$

Thus the equality $(*)$ holds also for $r \in R$ and consequently for any $t \in A(\mathcal{A})$.

We have a tbocs $\mathcal{A}^X = (R_X, W_X, \delta_X)$. Moreover there is a functor F^X : $\mathrm{Rep}\mathcal{A}^X \to \mathrm{Rep}\mathcal{A}$, such that for $M \in \mathrm{Rep}\mathcal{A}^X$, $F^X(M) = X \otimes_{R_X} M$ as R-modules and for $w \in W_0$, $w(x \otimes m) = \sum_i x_i \otimes \phi_{u_i,x}(w)m$. For $f = (f^0, f^1) : M \to N$ a morphism in $\mathrm{Rep}\mathcal{A}$, $F^X(f)$ is given for $x \otimes m \in X \otimes_{R_X} M, w \in W_1$ by:

$$F^X(f)^0(x \otimes m) = x \otimes f^0(m) + \sum_j p_j(x) \otimes f^1(\gamma_j)(m)$$

$$F^X(f)^1(w)(x \otimes m) = \sum_i f^1(u_i \otimes w \otimes x)(m).$$

Remark 4.1. *We recall from Proposition 5.3 of* [1] *that an object $L \in \mathrm{Rep}\mathcal{A}$ is isomorphic to some $F^X(M)$ iff ${}_RL \cong X \otimes_{R_X} L'$ as R-modules for some R_X-module L' . Observe that, in the above, if $\gamma \in T_R(W)$ is an element of degree 0 then $\gamma x \otimes m = \sum_i x_i \otimes \phi_{u_i,x}(\gamma)m$.*

If $(f, 0) : M \to N$ is a morphism in $\mathrm{Rep}\,\mathcal{A}^X$, then $F^X((f, 0)) = (g, 0)$. Consequently F^X induces a functor $F_0^X : \mathrm{Mod}\,A(\mathcal{A}^X) \to \mathrm{Mod}\,A(\mathcal{A})$ such that $F^X I_{\mathcal{A}^X} \cong I_{\mathcal{A}}F_0^X$. Here ${}_RF_0^X(M) \cong X \otimes_{R_X} M$, then F_0^X is a right exact functor which conmuts with arbitrary direct sums, then $F_0^X \cong Y \otimes_{A(\mathcal{A}^X)} -$ with Y the $A(\mathcal{A}) - A(\mathcal{A}^X)$-bimodule $F_0^X(A(\mathcal{A}^X))$. Thus ${}_RY \cong X \otimes_{R_X} A(\mathcal{A}^X)$ which is a finitely generated projective right $A(\mathcal{A}^X)$-module. Thus Y is an $A(\mathcal{A}) - A(\mathcal{A}^X)$-bimodule projective finitely generated on the right side.

Proposition 4.2. *Suppose $\mathcal{A} = (R, W, \delta)$ is a weak triangular tbocs, then $\mathcal{A}^X = (R_X, W_X; \delta_X)$ is a weak triangular tbocs.*

Proof. Consider $W_0^0 \subset ... \subset W_0^{r_0} = W_0$ and $(W_1)_0 \subset ... \subset W_1^{r_1} = W_1$ the corresponding filtrations given by the triangularity of $\mathcal{A}$.

We denote by $B_s(i, v, j)$ the R_X-bimodule generated by the elements of the form $f \otimes w \otimes x$ with $f \in X_i^*, w \in W_s^v, x \in X_j$.

We define

$$(W_X)_0^m = \sum_{i+2lv+j \leq m} B_0(i, v, j),$$

$$(W_X)_1^{m+l} = \sum_{i+2lv+j\leq m} B_1(i,v,j) \oplus \mathcal{R}^*,$$

$$(W_X)_1^i = \mathcal{R}_i^* \quad \text{for} \quad i \leq l.$$

As in [1] one can see, that $\mathcal{A}^X = (R_X, W_X, \delta_X)$ is a weak triangular tbocs with filtrations

$$0 = (W_X)_0^0 \subset ... \subset (W_X)_0^{2l(1+r_0)} = (W_X)_0$$

$$0 = (W_X)_1^0 \subset ... \subset (W_X)_1^{2l(1+r_1)+l} = (W_X)_1.$$

□

In the rest of this section we see a very useful reduction functor introduced originally in [7]. For this, let $\mathcal{A} = (R, W, \delta)$ be a tbocs with R a minimal k-algebra. Suppose $1 = \sum_{i=1}^n e_i$ is a decomposition into central primitive orthogonal idempotents, and $e_iR = k[x]_{f_i(x)}$ for $i = 1, .., t$, $e_jR = k$ for $j = t+1, ..., n$,

Now fix a natural number d and elements $g_1, ..., g_t \in k[x]$, with $(g_i, f_i) = 1$ for $i = 1, ..., t$.

For p a monic irreducible factor of g_i, $1 \leq i \leq t$ we put $Z_i(p) = e_iR/(p) \oplus ... \oplus e_iR/(p^d)$. For $1 \leq i \leq t$ we put $Z_i = \oplus_{p\in I(g_i)} Z_i(p)$, where $I(g_i)$ is the set of monic irreducible factors of g_i. For $i = t+1, ..., t+n$ we put $Z_i = e_iR = e_ik$. The R-module $Z = \oplus_i Z_i$ is basic with $\mathrm{End}_R^{op}(Z) = S_Z \oplus \mathcal{R}$ and $\mathcal{R} = \mathrm{radEnd}_R^{op}(Z)$.

We consider now $R' = (e_1R)_{g_1} \times ... \times (e_tR)_{g_t}$, clearly we have an epimorphism in the category of rings $R \to R'$ and $\mathrm{Hom}_R(Z, R') = 0$, $\mathrm{Hom}_R(R', Z) = 0$. Then if $X = Z \oplus R'$, we have a full and faithful functor:

$$F^X : \mathrm{Rep}\mathcal{A}^X \to \mathrm{Rep}\mathcal{A},$$

with $\mathcal{A}^X = (R_X, W_X, \delta_X)$ and $R_X = S_Z \times R'$.

The decomposition of Z into the direct sum of indecomposable R-modules of the form $(e_iR)/(p^u)$ with $1 \leq i \leq t$ and e_iR with $i > t$, and the decomposition of R' into the direct sum of R-modules of the form $(e_iR)_{g_i}$, with $1 \leq i \leq t$, gives a decomposition of R' into the direct sum of R-modules X_j. For each X_j we have the idempotent $e(X_j)$ which is the composition of the projection of X on X_j with the corresponding canonical inclusion in X.

For $1 \leq i \leq t$ and $1 \leq u \leq d$ we put $e_i^u(p) = e((e_iR)/(p^u))$, for p monic irreducible factor of g_i, and $e_i^0 = e((e_iR)_{g_i})$. For $t+1 \leq i \leq t+n$ we put $\underline{e}_i = e(e_iR)$.

The identity 1_X of R_X has the following decomposition into central primitive orthogonal idempotents:

$$1_X = \sum_{i=1}^t e_i^0 + \sum_{i=1}^t \sum_{p\in I(g_i)} \sum_{u=1}^d e_i^u(p) + \sum_{i=t+1}^{t+n} \underline{e}_i.$$

We have $e_i^0 R_X = (e_iR_X)_{g_i}$ for $1 \leq i \leq t$; $e_i^u(p)R_X = ke_i^u(p)$ for $1 \leq i \leq t$; $\underline{e}_iR_X = k\underline{e}_i$, for $t+1 \leq i \leq t+n$. Therefore R_X is a minimal k-algebra.

We recall that $(W_X)_0 = X^* \otimes_R W_0 \otimes_R X$. For $1 \leq i, j \leq t$ we have:

(1) $e_i^0(W_X)_0e_j^0 = (e_iR)_{g_i} \otimes_R e_iW_0e_j \otimes_R (e_jR)_{g_j}$;

(2) $e_i^0(W_X)_0e_j^u(p) = (e_iR)_{g_i} \otimes_R e_iW_oe_j \otimes_R (e_jR)/(p^u)$;

(3) $e_i^u(p)(W_X)_0e_j^0 = (e_iR)/(p^u))^* \otimes_R e_iW_oe_j \otimes_R (e_jR)g_j$;

(4) $e_i^u(p)(W_X)_0e_j^v(q) = (e_iR)/(p^u))^* \otimes_R e_iW_oe_j \otimes_R (e_jR)/(q^v)$.

For $1 \leq i \leq t; t+1 \leq j \leq t+n$ we have :

(5) $e_i^0(W_X)_0\underline{e}_j \cong (e_iR)_{g_i} \otimes_R e_iW_0e_j$;

(6) $\underline{e}_j(W_X)_0)e_i^0 \cong e_jW_0e_i \otimes_R (e_iR)_{g_i}$;
(7) $e_i^u(p)(W_X)_0)\underline{e}_j \cong (e_iR/(p^u))^* \otimes_R e_iW_0e_j$;
(8) $\underline{e}_j(W_X)_0)e_i^u(p) \cong e_jW_0e_i \otimes_R (e_iR/(p^u)$.
Finally for $t+1 \leq i \leq n$ we obtain:
(9) $\underline{e}_i(W_X)_0\underline{e}_j \cong e_iW_0e_j$.

The reduction functor F^X : Rep$\mathcal{A}^X$ $\rightarrow$ Rep$\mathcal{A}$ will be called a $(d, g_1, ..., g_t)$-unravelling.

Definition 4.3. *For $\mathcal{A} = (R, W, \delta)$ a tbocs, an object $M \in$ Rep$\mathcal{A}$ is an $R-E$-bimodule with $E = \text{End}_{\mathcal{A}}(M)^{op}$ and the right action of E on M given by $m.f = f^0(m)$ for $m \in M, f = (f^0, f^1) \in E$. Then M is called endofinite if the length of M as right E-module is finite, we will denote by* endol$M$ *the length of M as right E-module.*

Suppose now that M is an endofinite object in Rep$\mathcal{A}$. Then if $1 = \sum_i e_i$ is a decomposition into central primitive orhogonal idempotents of R, each e_iM is a $R-E$-bimodule and $M = \oplus_i e_iM$ as $R-E$-bimodules, thus endol$M = \sum_i \text{length}(e_iM_E)$.

Assume that $e_iR = R_i = k[x]_h$, then $E \subset \text{End}_{R_i}(e_iM) = E_i$. Then the length$(e_iM))_{E_i} \leq$ length$((e_iM)_E)$. Thus if M is endofinite, e_iM is a endofinite R_i-module. Therefore $e_iM_{Ri} \cong \sum_{j\in J} L_j$ with L_j indecomposable R_i-modules and in the set $\{L_j\}$ there are only a finite number of isomorphism classes. The only endofinite indecomposables R_i-modules are $k(x)$ and $k[x]/(x-\lambda)^m$ with $\lambda \in S(R_i)$, here $m \leq$ endol$M$.

Lemma 4.4. *If F^X : Rep$\mathcal{A}^X$ $\rightarrow$ Rep$\mathcal{A}$ is a $(d, g_1, ..., g_t)$ unravelling, for each endofinite object $N \in$ Rep$\mathcal{A}$ with* endol$N \leq d$*, there is a $M \in$ Rep$\mathcal{A}^X$ endofinite with* endol$M \leq$ endol$N$ *and $F(M) \cong N$.*

Proof. From the above considerations it follows that for $N \in$ Rep$\mathcal{A}$ with endol$N \leq d$, there is a $M \in$ Rep$\mathcal{A}^X$ with $F(M) \cong N$. We will assume that $F(M) = N$. Take $E_M = \text{End}_{\mathcal{A}^X}(M)^{op}$ and $E_N = \text{End}_{\mathcal{A}}(N)^{op}$. There is an isomorphism of k-algebras $\phi : E_M \rightarrow E_N$ induced by the functor F^X. Take $\mathcal{R} = \text{radEnd}_R(X)^{op}$ and an integer l with $\mathcal{R}^l = 0$.

We have a filtration $\mathcal{F}$ of R-modules of $X \otimes_{R_X} M = N$:

$$N_{l-1} = \mathcal{R}^{l-1}X \otimes_{R_X} M \subset ... \subset N_1 = \mathcal{R}X \otimes_{R_X} M \subset N_0 = X \otimes_{R_X} M.$$

Clearly $\mathcal{F}$ is a filtration of R-modules. The ring E_M also acts on N by $f(x \otimes n) = x \otimes nf = x \otimes f^0(n)$ for $f = (f^0, f^1) \in E_N$. The filtration $\mathcal{F}$ is also a filtration of $R-E_N$-bimodules. Now observe that for $n \in N_{l-1}, f \in E_N$, we have $nf = n\phi(f)$. The same happen for $\underline{n} \in N_i/N_{i+1}$ for $i = 0, ..., l-2$. Then the E_N length of N is equal to the length of N as E_M-module. Now we recall that there is a decomposition $X = \oplus_{i=1}^s X_i$ with the X_i indecomposables pairwise nonisomorphic. Take f_i the composition of the projection on the i-th summand followed of the corresponding injection. Then we have $1_X = \sum_{i=1}^s f_i$ a decomposition into primitive orthogonal idempotents, $Xf_i = X_i$. Here we have that X is projective finitely generated as right R_X-module, then each X_i is R_X projective, then $X_i \cong n_if_iR_X$ and $n_i \neq 0$. Then

$$\text{endol}N = \text{length}_{E_M} N = \text{length}_{E_M} X \otimes_{R_X} M = \sum_{i=1}^s \text{length}_{E_M} n_if_iM$$

$$\geq \sum_{i=1}^{s} \text{length}_{E_M} f_i M = \text{length}_{E_M} M = \text{endol} M.$$

This proves our claim. □

Definition 4.5. *Let R be a minimal k-algebra. Suppose $1 = \sum_{i=1}^{n} e_i$ is a decomposition into central primitive orthogonal idempotents, and $e_i R = k[x]_{f_i(x)}$ for $i = 1, .., t$, $e_j R = k$ for $j = t+1, ..., n$, we say that a R-bimodule U is thin if $e_i U e_j = 0$ for $i \leq t$ and $j \leq t$. A tbocs $\mathcal{A} = (R, W, \delta)$ is called thin if W_0 is a thin R-bimodule.*

Observe that having in account the above relations 1-9, if $\mathcal{A}$ is a thin tbocs, and $F^X : \text{Rep}\mathcal{A}^X \to \text{Rep}\mathcal{A}$ is a $(d, g_1, ..., g_t)$-unravelling, then $\mathcal{A}^X$ is also a thin tbocs.

Let S be a k-subalgebra of R, we recall that U a R-bimodule is called S- free if there is a S-subimodule $\hat{U}$ of U such that the morphism of R-bimodules $\mu_U : R \otimes_S \hat{U} \otimes_S R \to U$ given by $\mu_U(r_1 \otimes u \otimes r_2) = r_1 u r_2$ is an isomorphism.

Lemma 4.6. *Suppose U is a thin R-bimodule, then U is S-free if for all $1 \leq i \leq t$, Ue_i is free as right $e_i R$-module and $e_i U$ is free as left $e_i R$-module.*

Proof. Observe that Ue_i is free as right $e_i R$-module iff it is S free as R-bimodule. Similarly $e_i U$ is free as left $e_i R$-module iff it is S-free as a R-bimodule. Therefore if the hypothesis of the proposition holds, then for each $1 \leq i \leq t$ there are S-subbimodules V_i of Ue_i and ${}_iV$ of $e_i U$, such that the morphisms: $\mu_{V_i} : R \otimes_S V_i \otimes_S R \to Ue_i$ and $\mu : R \otimes_S ({}_iV) \otimes_S R \to e_i U$ are isomorphisms.

For $V_0 = \sum_{i,j \geq t+1} e_i U e_j$, the morphism $\mu_{V_0} : R \otimes_S V_0 \otimes_S R \to \sum_{i,j \geq t+1} e_i U e_j$ is clearly an isomorphism. Consequently, if $V = \sum_i (V_i + {}_i V) + V_0$, then the morphism $\mu_V : R \otimes_S V \otimes_S R \to U$, is an isomorphim. Therefore V is a S-free generator for the R-bimodule U. □

Definition 4.7. *Let U be a R-bimodule, a filtration $U^1 \subset ... \subset U^r = U$ is called a S-free filtration if for $u = 1, ..., r$ there are S-free generators V^u of U^u such that $V^1 \subset ... \subset V^r$.*

The following is clear.

Lemma 4.8. *Let U be a thin R-bimodule, suppose that for $1 \leq i \leq t$ there are S-free filtrations $U_i^1 \subset ...U_i^r = Ue_i$, ${}_iU^1 \subset ... \subset_i U^r = e_i U$, and $U_0^1 \subset ... \subset U_0^r = \sum_{i,j \geq t+1} e_i U e_j$, then if for $1 \leq u \leq r$, $U^u = \sum_{i \leq t} (U_i^u +_i U^u) + U_0^u$,*

$$U^1 \subset ... \subset U^r = U$$

is a S-free filtration for U.

Proposition 4.9. *Let $\mathcal{A} = (R, W, \delta)$ be a thin weak triangular tbocs, then there is a $(d, g_1, ..., g_t)$- unravelling,*

$$F^X : \text{Rep}\mathcal{A}^X \to \text{Rep}\mathcal{A}$$

such that $\mathcal{A}^X$ is a thin triangular tbocs.

Proof. Here $\mathcal{A}$ is weak triangular, we have a filtration

$$w : \quad 0 = W_0^0 \subset W_0^1 \subset ... \subset W_0^r = W_0$$

satisfying the condition T.1 of Definition 3.2. There are elements $g_1, ..., g_t$ such that for $1 \leq i \leq t, 1 \leq u \leq r$, $(e_iR)_{g_i} \otimes_R W_0^u$ and $W_0^u \otimes_R (e_iR)_{g_i}$ are free left $(e_iR)_{g_i}$-modules and free right $(e_iR)_{g_i}$-modules respectively, and for $1 \leq u \leq r-1$, $(e_iR)_{g_i} \otimes_R W_0^{u-1}$ is a direct summand as left $(e_iR)_{g_i}$ -module of $(e_iR)_{g_i} \otimes_R W_0^u$ and $W_0^{u-1} \otimes_R (e_iR)_{g_i}$ is a summand as right $(e_iR)_{g_i}$-module of $W_0^u \otimes_R (e_iR)_{g_i}$.

Now $S = S_0 \times S_1$ with $S_0 = \sum_{i>t} e_ik$ and $S_1 = \sum_{i \leq t} e_ik$. Here W_0 is thin, $S_1W_0^u \otimes_R (e_iR)_{g_i} = 0$ and $(e_iR)_{g_i} \otimes_R W_0^uS_1 = 0$. Thus each $W_0^u \otimes_R (e_iR)_{g_i}$ is a $S_0 - (e_iR)_{g_i}$-bimodule, therefore there are S_0-left modules $\hat{W}_i^u$-submodules of $W_0^u \otimes_R (e_iR)_{g_i}$) such that, $\hat{W}_i^{u-1} \subset \hat{W}_i^u$ and the morphisms

$$\mu_{i,u} : \hat{W}_i^u \otimes_k (e_iR)_{g_i} \to W_0^u \otimes_R (e_iR)_{g_i}, \quad \mu_{i,u}(w \otimes f) = wf,$$

are isomorphisms. Similarly, there is a S_0-right submodule ${}_i\hat{W}^u$ of $(e_iR)_{g_i} \otimes_R W_0^u$ such that ${}_i\hat{W}^{u-1} \subset_i \hat{W}^u$ and

$$\nu_{i,u} : (e_iR)_{g_i} \otimes_k {}_i\hat{W}^u \to (e_iR)_{g_i} \otimes_R W_0^u, \quad \nu_{i,u}(f \otimes w) = fw,$$

is an isomorphism.

Take now the $(d, g_1, ..., g_t)$-unravelling, F^X : Rep$\mathcal{A}^X \to$ Rep$\mathcal{A}$. Then there is a filtration of $(W_X)_0$:

$$0 = (W_X)_0^0 \subset (W_X)_0^1 \subset ... \subset (W_X)_0^{2(r+1)} = (W_X)_0$$

having condition T.1 of Definition 3.2.

We define:

$$(S_X)_0 = \sum_{i>t} \underline{e_i}k, \ (S_X)_1 = \sum_{i \leq t} e_i^0k, \ (S_X)_2 = \sum_{i \leq t} \sum_{p \in I(g_i)} \sum_{u=1}^{t} e_i^u(p)k.$$

Then we have $S_X = (S_X)_0 \times (S_X)_1 \times (S_X)_2$, $(S_X) \cong S_0$, $(S_X)_1 \cong S_1$ and $R_X = (S_X)_0 \times (S_X)_2 \times R'$ with $(S_X)_1 \subset R' = \sum_{i \leq t} e_i^0R_X$.

Each $W_0^u \otimes_R (e_iR)_{g_i}$ is a $S_0 - (e_iR)_{g_i}$-bimodule.

Through the projection $R_X \to (S_X)_0$ followed by the isomorphism $(S_X)_0 \to S_0$ and the projection $R_X \to (e_iR)_{g_i}$, $W_0^u \otimes_R (e_iR)_{g_i}$ becomes a R_X-bimodule.

Moreover we have the commutative diagram:

$$\begin{array}{ccc} \hat{W}_i^u \otimes_k (e_iR)_{g_i} & \xrightarrow{\mu_{i,u}} & W_0^u \otimes_R (e_iR)_{g_i} \\ \cong\downarrow & & \downarrow= \\ R_X \otimes_{S_X} \hat{W}_i^u \otimes_{S_X} R_X & \xrightarrow{\mu_{W_0^u}} & W_0^u \otimes_R (e_iR)_{g_i}, \end{array}$$

therefore $\hat{W}_i^u$ is a S_X-free generator of the R_X-bimodule $W_0^u \otimes_R (e_iR)_{g_i}$.

For $2l(s+1) \leq m \leq 2l(s+2) - 1$ there is an isomorphism of R_X-bimodules:

$$(W_X)_0^m e_i^0 \overset{\phi_m}{\to} (W_0^s e_i) \otimes_R (e_iR)_{g_i}.$$

Then $V_i^m := \phi_m^{-1}(\hat{W}_i^s)$ is a S_X-free generator of $(W_X)_0^m e_i^0$.

We have the following commutativity diagram:

$$\begin{array}{ccc} (W_X)_0^m e_i^0 & \xrightarrow{\phi_m} & (W_0^s e_i) \otimes_R (e_iR)_{g_i} \\ \downarrow & & \downarrow \\ (W_X)_0^{m+1} e_i^0 & \xrightarrow{\phi_{m+1}} & (W_0^{s'} e_i) \otimes_R (e_iR)_{g_i}, \end{array}$$

with $s' = s+1$ if $m = 2l(s+2)-1$ and $s' = s$ otherwise. Thus we have $V_i^m \subset V_i^{m+1}$, and consequently the filtration

$$(W_X)_0^1 e_i^0 \subset .... \subset (W_X)_0^{2l(r+1)} e_i^0 = (W_X)_0 e_i^0$$

is a S_X-free filtration. In a similar way one can prove that the filtration

$$e_i^0 (W_X)_0^1 \subset .... \subset e_i^0 (W_X)_0^{2l(r+1)} = e_i^0 (W_X)_0,$$

is also a S_X-free filtration. Therefore by Lemma 4.8 the filtration w is a S_X-free filtration. Clearly $(W_X)_1$ is a S_X-free R-bimodule, therefore our tbocs $\mathcal{A}^X$ is free triangular. □

Proposition 4.10. *Let $\mathcal{A} = (R, W, \delta)$ be a thin free triangular tbocs, which is not of wild representation type, then given a natural number d, there is a finite set of full and faithful functors $F_i : \mathrm{Rep}\,\mathcal{B}_i \to \mathrm{Rep}\,\mathcal{A}$, $i = 1, ..., m$ such that:*
i) each $\mathcal{B}_i = (R_i, W^i, \delta_i)$ is a minimal triangular tbocs;
ii) for $M \in \mathrm{Rep}\,\mathcal{A}$ with $\mathrm{endol} M \leq d$, there is an $i \in \{1, ..., m\}$ and $N \in \mathrm{Rep}\,\mathcal{B}_i$ with $F_i(N) \cong M$;
iii) for each $i \in \{1, ..., m\}$ there is a $A(\mathcal{A}) - R_i$-bimodule Y_i, projective finitely generated over the right side such that

$$F_i I_{\mathcal{B}_i} \cong I_{\mathcal{A}}(Y_i \otimes_{R_i} -).$$

Proof. By Proposition 4.9 there is a functor $F^X : \mathrm{Rep}\,\mathcal{A}^X \to \mathrm{Rep}\,\mathcal{A}$, given by a $(d, g_1, ..., g_t)$-unravelling such that $\mathcal{A}^X$ is a free triangular tbocs. Moreover for M with $\mathrm{endol} M \leq d$ there is a $N \in \mathrm{Rep}\,\mathcal{A}^X$ with $F^X(N) \cong M$. Since $\mathcal{A}$ is not of wild representation type then $\mathcal{A}^X$ is not of wild representation type. Therefore by [8] or by Theorem 11.1 of [4] there is a finite set of full and faithful functors $G_i : \mathrm{Rep}\,\mathcal{B}_i \to \mathrm{Rep}\,\mathcal{A}^X$ satisfying conditions i), ii) and iii). Then using Lemma 4.4 and the second part of Remark 4.1 the full and faithful functors $F_i = F^X G_i : \mathrm{Rep}\,\mathcal{B}_i \to \mathrm{Rep}\,\mathcal{A}$ satisfy i), ii) and iii). □

Remark 4.11. *With the notation of Proposition 4.10 suppose $1_R = \sum_{i=1}^s e_i$ is a decomposition into central primitive orthogonal idempotents. We consider $D(\mathcal{A}) = \mathbb{Q}^s$, for $M \in \mathrm{rep}\mathcal{A}$ we put $\underline{\dim} M = (\dim_k e_1 M, ..., \dim_k e_s M)$.*

For $i = 1, ..., t$, R_i is a minimal k-algebra thus we have a decomposition of $1_{R_i} = \sum_j^{s(j)} f_{i,j}$ with $f_{i,j}, j = 1, ..., s(j)$ a set of central primitive orthogonal idempotents.

The functor $F_i : \mathrm{Rep}\mathcal{B}_i \to \mathrm{Rep}\mathcal{A}$ determines a k-linear map $t_{F_i} : D(\mathcal{B}_i) \to D(\mathcal{A})$ such that for $M \in \mathrm{rep}\mathcal{B}_i$ we have $\underline{\dim} F_i(M) = t_{F_i}(\underline{\dim} M)$.

5. A category of morphisms

Let $\mathcal{A} = (R, W, \delta)$ be a minimal triangular tbocs. Supose $1_R = \sum_{j=1}^n e_j$ with $\{e_j\}_{j=1}^n$ central primitive orthogonal idempotents in R, now assume that $e = \sum_j^t$ with $t < n$ is such that $eR = Re = eRe$ is a semisimple k-algebra, we denote $f = \sum_{j>t} e_j$. From the triangularity condition $T.3$ of Definition 3.2 we have a filtration $0 \subset W^1 \subset .... \subset W^m = W$.

We will consider the following category of radical morphisms in $\mathrm{Rep}\mathcal{A}$, $\mathcal{M}$.

The objects of $\mathcal{M}$ are the radical morphisms $\phi : X \to Y$ with $fX = 0$. The morphisms from $\phi : X \to Y$ to $\phi' : X' \to Y'$ two objects of $\mathcal{M}$, are given by pairs of morphisms $u = (u_1, u_2)$, $u_1 : X \to X'$, $u_2 : Y \to Y'$, morphisms in Rep$\mathcal{A}$ such that $u_2\phi = \phi u_1$.

If $v = (v_1, v_2)$ is a morphism from $\phi' : X' \to Y'$ to $\phi'' : X'' \to Y''$, then $vu = (v_1u_1, v_2u_2)$. Observe that if $\phi : X \to Y$ is a morphism object of $\mathcal{M}$, then this morphism has the form $\phi = (0, \phi^1)$.

Clearly $\mathcal{M}$ is a category, we shall see that this category is equivalent to the category of representations of a triangular tbocs.

We first describe the morphisms in the category $\mathcal{A}$.

Suppose $u = (u_1, u_2) : \phi \to \phi'$ is a morphism in $\mathcal{M}$ with $\phi = (0, \phi^1) : X \to Y$, $\phi' = (0, (\phi')^1) : X' \to Y'$. Here $u_1 = (u_1^0, u_1^1), u_2 = (u_2^0, u_2^1)$, $u_2\phi = \phi' u_1$.

For $w \in W_1 = W$ with $\delta(w) = \sum_s w_s^1 \otimes w_s^2$ we have:

$$(\phi')^1(w)u_1^0 + \sum_s (\phi')^1(w_s^1)u_1^1(w_s^2) = u_2^0\phi^1(w) + \sum_s u_1^1(w_s^1)\phi^1(w_s^2).$$

For $w \in W$, $x \in X$,

$$\phi^1(wf)(x) = \phi^1(fx) = 0, \quad \text{therefore} \quad \phi^1(w) = \phi^1(we).$$

In a similar way we have $(\phi')^1(w) = (\phi')^1(we)$. Moreover :

$$u_1^1(fw)(x) = fu_1^1(w)(x) = 0, u_1^1(wf)(x) = u_1^1(fx) = 0,$$

therefore $u_1^1(w) = u_1^1(ewe)$.

Then for $w \in W$ with $\delta(w) = \sum_s w_s^1 \otimes w_s^2$, we have:

$$(2) \quad (\phi')^1(we)u_1^0 - u_2^0\phi^1(we) = \sum_s u_1^1(w_s^1)\phi^1(w_s^2e) - \sum_s (\phi')^1(w_s^1e)u_1^1(ew_s^2e).$$

Now in order to describe the category $\mathcal{M}$ in terms of a tbocs we introduce the following triangular tbocs, $\mathcal{B} = (S, W_\mathcal{B}, \delta_\mathcal{B})$, with

$$S = \begin{pmatrix} R & 0 \\ 0 & eRe \end{pmatrix}, (W_\mathcal{B})_0 = \begin{pmatrix} 0 & We \\ 0 & 0 \end{pmatrix}, (W_\mathcal{B})_1 = \begin{pmatrix} W & 0 \\ 0 & eWe \end{pmatrix}.$$

For $w \in W$ with $\delta(w) = \sum_s w_s^1 \otimes w_s^2$ we put

$$\delta_\mathcal{B}\begin{pmatrix} 0 & we \\ 0 & 0 \end{pmatrix} = \sum_s \begin{pmatrix} 0 & w_s^1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & w_s^2e \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & w_s^1e \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & ew_s^2e \end{pmatrix}$$
$$= \sum_s \begin{pmatrix} 0 & w_s^1 \otimes w_s^2e - w_s^1e \otimes ew_s^2e \\ 0 & 0 \end{pmatrix}.$$
$$\delta_\mathcal{B}\begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} w_s^1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} w_s^2 & 0 \\ 0 & 0 \end{pmatrix} = \sum_s \begin{pmatrix} w_s^1 \otimes w_s^2 & 0 \\ 0 & 0 \end{pmatrix},$$
$$\delta_\mathcal{B}\begin{pmatrix} 0 & 0 \\ 0 & ewe \end{pmatrix} = \sum s \begin{pmatrix} 0 & 0 \\ 0 & ew_s^1e \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & ew_s^2e \end{pmatrix}$$
$$= \sum_s \begin{pmatrix} 0 & 0 \\ 0 & ew_s^1e \otimes ew_s^2e \end{pmatrix},$$

using Leibnitz rule one can extend $\delta_\mathcal{B}$ to a function $\delta_\mathcal{B} : T_R(W) \to T_R(W)$, in order to see that $\delta_\mathcal{B}^2 = 0$, it is enough to prove that for $w \in W$ we have:

$$\delta_\mathcal{B}^2\begin{pmatrix} 0 & we \\ 0 & 0 \end{pmatrix} = 0, \ \delta_\mathcal{B}^2\begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix} = 0, \ \delta_\mathcal{B}^2\begin{pmatrix} 0 & 0 \\ 0 & ewe \end{pmatrix} = 0.$$

Take $w \in W$ with $\delta(w) = \sum_s w_s^1 \otimes w_s^2$ and $\delta(w_s^1) = \sum_j w_{s,j}^{1,1} \otimes w_{s,j}^{1,2}$, $\delta(w_s^2) = \sum_j w_{s,j}^{2,1} \otimes w_{s,j}^{2,2}$. From $\delta^2 = 0$ we obtain:

$$(1) \qquad \sum_{s,j} w_{s,j}^{1,1} \otimes w_{s,j}^{1,2} \otimes w_s^2 - \sum_{s,j} w_s^1 \otimes w_{s,j}^{2,1} \otimes w_{s,j}^{2,2} = 0.$$

Taking $\delta_{\mathcal{B}}^2 \begin{pmatrix} 0 & we \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}$, we have:

$$u = \sum_{s,j} w_{s,j}^{1,1} \otimes w_{s,j}^{1,2} \otimes w_s^2 e - \sum_{s,j} w_s^1 \otimes w_{s,j}^{2,1} \otimes w_{s,j}^{2,2} e$$
$$+ \sum_{s,j} w_{s,j}^{1,1} \otimes w_{s,j}^{1,2} e \otimes e w_s^2 e - \sum_{s,j} w_s^1 \otimes w_{s,j}^{2,1} e \otimes e w_{s,j}^{2,2} e$$
$$+ \sum_{s,j} w_{s,j}^{1,1} e \otimes e w_{s,j}^{1,2} e \otimes e w_s^2 e - \sum_{s,j} w_s^1 e \otimes e w_{s,j}^{2,1} e \otimes e w_{s,j}^{2,2} e.$$

Now taking the projections $W \otimes_R W \otimes_R W \otimes_R W \to W \otimes_R W \otimes_R W \otimes_R We$, given by $w_1 \otimes w_2 \otimes w_3 \to w_1 \otimes w_2 \otimes w_3 e$; $W \otimes_R W \otimes_R W \otimes_R W \to W \otimes_R W \otimes_R We \otimes_R eWe$ given by $w_1 \otimes w_2 \otimes w_3 \to w_1 \otimes w_2 e \otimes e w_3 e$ and $W \otimes_R W \otimes_R W \otimes_R W \to We \otimes_R eWe \otimes_R eWe \otimes_R eWe$ given by $w_1 \otimes w_2 \otimes w_3 \to w_1 e \otimes e w_2 e \otimes e w_3 e$ of (1) we obtain that $u = 0$.

In a similar way we obtain the second and thirth equalities.

Proposition 5.1. *The tbocs $\mathcal{B} = (S, W_{\mathcal{B}}, \delta_{\mathcal{B}})$ is a weak thin triangular tbocs.*

Proof. Here $\mathcal{A} = (R, W, \delta)$ is triangular, by definition there is a basic semisimple k-subalgebra R_0 of R. Then $S_0 = \begin{pmatrix} R_0 & 0 \\ 0 & eR_0 e \end{pmatrix}$ is a basic semisimple k-subalgebra of S. We have filtrations $\{0\} \subset (W_{\mathcal{B}})_i^1 \subset (W_{\mathcal{B}})_i^1 \subset \ldots \subset (W_{\mathcal{B}})_i^m = (W_{\mathcal{B}})_i$, for $i = 0, 1$, with

$$(W_{\mathcal{B}})_0^i = \begin{pmatrix} 0 & W^i e \\ 0 & 0 \end{pmatrix}, (W_{\mathcal{B}})_1^i = \begin{pmatrix} W^i & 0 \\ 0 & eW^i e \end{pmatrix}.$$

Then $\mathcal{B}$ satisfies condition $T.1$, and $T.3$ of Definition 3.2. Now there is a $R_0 - R_0$ subimodule $\hat{W}$ of W such that $W \cong R \otimes_{R_0} \hat{W} \otimes_{R_0} R$. Then $eWe \cong eRe \otimes_{eR_0 e} e\hat{W}e \otimes_{eR_0 e} eRe$, therefore:

$$S \otimes_{S_0} \begin{pmatrix} \hat{W} & 0 \\ 0 & e\hat{W}e \end{pmatrix} \otimes_{S_0} S \cong \begin{pmatrix} W & 0 \\ 0 & eWe \end{pmatrix}.$$

Thus we also have condition $T.4$ of Definition 2.1. This proves our result.

Theorem 5.2. *There exists a functor $F : \mathrm{Rep}\mathcal{B} \to \mathcal{M}$ which is an equivalence of categories.*

Proof. We have $A(\mathcal{B}) = T_S((W_{\mathcal{B}})_0) = \begin{pmatrix} R & We \\ 0 & eRe \end{pmatrix}$. We have in $A(\mathcal{B})$ the idempotents $\eta = \begin{pmatrix} 1_R & 0 \\ 0 & 0 \end{pmatrix}$, $\sigma = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}$. Take $V \in \mathrm{Rep}\mathcal{B}$, here V is an $A(\mathcal{B})$-module then $V = \eta V \oplus \sigma V$ as k-modules. Here $V_1 = \eta V$ is a R-module and $V_2 = \sigma V$ is a eRe-module. The action of $A(\mathcal{B})$ on V induces a morphism of R-modules: $h : We \otimes_{eRe} V_2 \to V_1$. Conversely if V_1 is a R-module, V_2 is a eRe-module and $h : We\otimes_{eRe} V_2 \to V_1$ a morphism of R-modules the triple $(V_1, V_2; h)$ determines an $A(\mathcal{B})$-module V.

We recall we have an isomorphism

$$\psi : \mathrm{Hom}_R(We \otimes_{eRe} V_2, V_1) \to \mathrm{Hom}_{R-eRe}(We, \mathrm{Hom}_k(V_2, V_1)).$$

Then if $V \in \mathrm{Rep}\,\mathcal{B}$ is given by the triple $(V_1, V_2; h)$ we define $F(V) = \phi = (0, \phi^1) : V_2 \to V_1$ with $\phi^1 = \psi(h)\tau \in \mathrm{Hom}_{R-eRe}(We, \mathrm{Hom}_k(V_2, V_1))$ $= \mathrm{Hom}_{R-R}(We, \mathrm{Hom}_k(V_2, V_1))$, where τ is the inclusion of We in W. Clearly ϕ is a morphism in $\mathcal{A}$ which is an object in $\mathcal{M}$.

Now take $z : V \to V'$ a morphism in $\mathrm{Rep}\,\mathcal{B}$, $z = (z^0, z^1)$. Here z^0 is a morphism of S-modules from V to V', then $z^0 = (z_1^0, z_2^0)$ with $z_1^0 : V_1 \to V_2$ a morphism of R-modules and $z_2^0 : V_2 \to V_2'$ a morphism of eRe-modules. On the other hand:

$$z^1 : \begin{pmatrix} W & 0 \\ 0 & eWe \end{pmatrix} \to \mathrm{Hom}_k(V, V')$$

is a morphism of $S-S$-bimodules, therefore $z^1 = (z_1^1, z_2^1)$ with $z_1^1 : W \to \mathrm{Hom}_k(V_1, V_1')$ a morphism of $R-R$-bimodules and $z_2^1 : eWe \to \mathrm{Hom}_k(V_2, V_2')$ a morphism of $eRe-eRe$-bimodules. Since $z : V \to V'$ is a morphism in $\mathrm{Rep}\mathcal{B}$ we have for all $we \in We$ with $\delta(w) = \sum_s w_s^1 \otimes w_s^2$ and $v_1 \in V_1, v_2 \in V_2$:

$$\begin{pmatrix} 0 & we \\ 0 & 0 \end{pmatrix} z^0 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = z^0 \begin{pmatrix} 0 & we \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + z^1\delta_{\mathcal{B}} \begin{pmatrix} 0 & we \\ 0 & 0 \end{pmatrix}.$$

Then we obtain:

$$\begin{pmatrix} h'(w \otimes z_2^0(v_2)) \\ 0 \end{pmatrix} = z^0 \begin{pmatrix} h(w \otimes v_2) \\ 0 \end{pmatrix}$$

$$+\sum_s z^1 \left[\begin{pmatrix} w_s^1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & w_s^2 e \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & w_s^2 e \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & ew_s^2 e \end{pmatrix} \right] \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

from here we obtain the equality:

$$(3) \quad (\phi)^1(w)(z_2^0(v_2)) = z_1^0(\phi^1(w)(v_2))$$

$$+\sum_s z_1^1(w_s^1)(\phi^1(w_s^2)(v_2)) - \sum_s (\phi')^1(w_s^1 e)(z_2^1(ew_s^2 e)(v_2)).$$

We have that $u_1 = (z_1^0, z_1^1)$ is a morphism from V_1 to V_1' in $\mathrm{Rep}\mathcal{A}$, and $u_2 = (z_2^0, z_2^1)$ is a morphism from V_2 to V_2'. Then by (2) we have that $u = (u_1, u_2)$ is a morphism from $\phi = F(V)$ to $\phi' = F(V')$. We put $F(z) = u$. Now is clear that if $F(z) = 0$, then $z = 0$. Moreover for any morphism $u = (u_1, u_2) : \phi \to \phi'$ $u_1 = (u_1^0, u_1^1), u_2 = (u_2^0, u_2^1)$. Here $u_1^0 \in \mathrm{Hom}_R(V_1, V_1')$, $u_2^0 \in \mathrm{Hom}_{eRe}(V_2, V_2')$. Thus the pair (u_1^0, u_2^0) define a morphism of S-modules $z^0 : V \to V'$. In a similar way the pair of morphisms (u_1^1, u_2^1) define a morphism of $S-S$-bimodules $z^1 : \begin{pmatrix} W & 0 \\ 0 & eWe \end{pmatrix} \to \mathrm{Hom}_k(V, V')$. Thus we obtain a morphism $z = (z^0, z^1) : V \to V'$ in $\mathrm{Rep}\mathcal{B}$ such that $F(z) = u$.

Now if $z : V \to V'$ and $z' : V' \to V''$ are morphisms then $F(z')F(z) = F(z'z)$. Clearly F sends identities into identities and F is a dense functor, this proves our claim. □

6. Main Results

This section is devoted to the proofs of Theorem 1.1 and Theorem 1.2. In the following for P a projective Λ-module we denote by $S(P)$ the complex with $S(P)^1 = P$ and $S(P)^i = 0$ for $i \neq 1$. For $h : P \to P'$ a morphism of Λ-modules we denote by $S(h) : S(P) \to S(P')$ the morphism of complexes given by $S(h)^1 = h$, $S(h)^i = 0$ for $i \neq 1$. For $n \geq 1$, we consider the following category $\mathcal{M}_n$ of morphisms in $\mathbf{C^1_n}(\mathrm{Proj}\,\Lambda)$. The objects of $\mathcal{M}_n$ are radical morphisms $f : S(P) \to X$ in $\mathbf{C^1_n}(\mathrm{Proj}\,\Lambda)$ with P a projective Λ-module and X any object in $\mathbf{C^1_n}(\mathrm{Proj}\,\Lambda)$. The morphisms from $f : S(P) \to X$ to $f' : S(P') \to X'$ are given by pairs of morphisms $u = (u_1, u_2)$, $u_1 : P \to P'$, $u_2 : X \to X'$ such that $u_2 f = f' S(u_1)$. If $u = (u_1, u_2)$ is a morphism from $f : S(P) \to X$ to $f' : S(P') \to X'$ and $v = (v_1, v_2)$ is a morphism from $f' : S(P') \to X'$ to $f'' : S(P'') \to X''$, then $vu = (v_1u_1, v_2u_2)$. The identity morphism in the object $f : S(P) \to X$ is given by the pair (id_P, id_X).

Proposition 6.1. *There is a functor $G : \mathcal{M}_n \to \mathbf{C^1_{n+1}}(\mathrm{Proj}\,\Lambda)$ which is an equivalence of categories.*

Proof. Take $f : S(P) \to X$ an object in $\mathcal{M}_n$. We have the morphism $f^1 : P \to X^1$, f is a radical morphism, thus $\mathrm{Im} f^1 \subset \mathrm{rad} X^1$, moreover f is a morphism of complexes, we have $d^1_X f^1 = f^2 d^1_P = 0$. Therefore we have the complex $G(f)$ in $\mathbf{C^1_{n+1}}(\mathrm{Proj}\,\Lambda)$ given by $G(F)^i = 0$ for i outside the interval $[1, ..., n+1]$, $G(f)^1 = P$, $G(f)^{i+1} = X^i$ for $i = 1, ..., n$, $d^1_{G(f)} = f^1$, $d^{i+1}_{G(f)} = d^i_X$ for $i = 1, ..., n$.

Now if $u = (u_1, u_2)$ is a morphism from $f : S(P) \to X$ to $f' : S(P') \to X'$, we define $G(u)$ in the following way: $G(u)^i = 0$ for i outside the interval $[1, ..., n + 1]$, $G(u)^1 = u_1 : G(f)^1 = P \to G(f')^1 = P'$, $G(u)^{i+1} = u^i_2 : G(f)^{i+1} = X^i \to G(f')^{i+1} = (X')^i$ for $i = 1, ..., n$.

We have $d^1_{G(f)} G(u)^1 = (f')^1 u_1 = (u_2)^1 f' = G(u)^2 d^1_{G(f)}$. For $i = 1, ..., n$ we have $d^{i+1}_{G(f')} G(u)^{i+1} = d^i_{X'} u^i_2 = u^{i+1}_2 d^i_X = G(u)^{i+2} d^{i+1}_{G(f)}$. From here we conclude that $G(u) : G(f) \to G(f')$ is a morphism of complexes. We have $G(id_f) = id_{G(f)}$. Now if v is a morphism from $f' : S(P') \to X'$ to $f'' : S(P'') \to X''$, $G(v)G(u) = G(vu)$. Clearly G is a full, faithful dense functor. □

Definition 6.2. *Take $X \in \mathbf{C_n}(\mathrm{Proj}\,\Lambda)$. Then $E_X = \mathrm{End}_{\mathbf{C_n}(\mathrm{Proj}\,\Lambda)}(X)$ acts by the left on each X^i, we say that X has finite endolength if each X^i has finite length as E_X-left module. We define $\mathrm{endol}(X) = \sum_i \mathrm{length}_{E_X} X^i$.*

Now suppose $P_1, ..., P_m$ is a representative system of the isomorphism classes of the indecomposable projective Λ-modules. For H a Λ-module we put $\underline{\dim} H = (\dim_k \mathrm{Hom}(P_1, M), ..., \dim_k \mathrm{Hom}(P_m, M))$.

For the category $\mathbf{C_n}(\mathrm{proj}\,\Lambda)$ we consider $c(\mathbf{C_n}(\mathrm{proj}\,\Lambda)) = \mathbb{Q}^{nm}$. For $X \in \mathbf{C_n}(\mathrm{proj}\,\Lambda)$, we put $c(X) = (\underline{\dim} X_1/\mathrm{rad} X_1; ...; \underline{\dim} X_n/\mathrm{rad} X_n)$.

Let $\mathcal{C}$ be a k-category and E a k-algebra, a $\mathcal{C} - E$-object is an object $M \in \mathcal{C}$ endowed with a homomorphism of k-algebras $\alpha_M : E \to \mathrm{End}_{\mathcal{C}}(M)^{op}$. If M and N are $\mathcal{C} - E$-objects, a morphism of $\mathcal{C} - E$-objects from M to N is a morphism $f : M \to N$ in $\mathcal{C}$ such that for all $r \in E$, $f \alpha_M(r) = \alpha_N(r) f$. If $F : \mathcal{C} \to \mathcal{D}$ is a functor and M is a $\mathcal{C} - E$-object, then $F(M)$ is a $\mathcal{D} - E$-object, taking $\alpha_{F(M)}$ the composition $E \overset{\alpha_M}{\to} \mathrm{End}_{\mathcal{C}}(M)^{op} \overset{F}{\to} \mathrm{End}_{\mathcal{D}}(F(M))^{op}$. Clearly if $f : M \to N$ is a morphism of $\mathcal{C} - E$-objects, $F(f) : F(M) \to F(N)$ is a morphism of $\mathcal{D} - E$-objects.

Example 1

A $\mathbf{C_n}(\operatorname{Proj}\Lambda) - E$-object is a complex $X \in \mathbf{C_n}(\operatorname{Proj}\Lambda)$ such that each X^i is a $\Lambda - E$-bimodule and for all $i \in \mathbb{Z}$, d^i_X is a morphism of $\Lambda - E$-bimodules. If X, Y are $\mathbf{C_n}(\operatorname{Proj}\Lambda) - E$-objects, a morphism of complexes $f : X \to Y$ is a morphism of $\mathbf{C_n}(\operatorname{Proj}\Lambda) - E$-objects if each $f^i : X^i \to Y^i$ is a morphism of $\Lambda - E$-bimodules.

Example 2

Let $\mathcal{B}$ and $\mathcal{C}$ be full subcategories of a category $\mathcal{D}$, consider $\mathcal{M}$ the category of morphisms $f : X \to Y$ in $\mathcal{D}$ with $X \in \mathcal{B}, Y \in \mathcal{C}$. Then $f : X \to Y$ is a $\mathcal{M} - E$-object if f is a morphism of $\mathcal{D} - E$-objects. Clearly $u = (u_1, u_2) : (f : X \to Y) \to (f' : X' \to Y')$ is a morphism of $\mathcal{M} - E$-objects if and only if u_1 and u_2 are morphisms of $\mathcal{D} - E$-objects.

Example 3

Let $\mathcal{A} = (R, W, \delta)$ be a tbocs. We say that M is an $\mathcal{A} - E$-bimodule if it is a Rep$\mathcal{A} - E$-object. Then for $x \in E$ we have $\alpha_M(x) = (\alpha_M(x)^0, \alpha_M(x)^1)$. The $\mathcal{A} - E$-bimodule M is said to be proper if for all $x \in E$, $\alpha_M(r)^1 = 0$. In this case M is an $R - E$-bimodule with $mx = \alpha_M(x)^0(m)$. Moreover for $a \in A(\mathcal{A}), m \in M$, $(am)x = \alpha_M(x)^0(am) = a\alpha_M(x)^0(m) = a(mx)$, consequently M is a $A(\mathcal{A}) - E$-bimodule. Clearly if M is a $A(\mathcal{A}) - E$-bimodule then M is a proper $\mathcal{A} - E$-bimodule.

If $f = (f^0, f^1) : M \to N$ is a morphism in Rep$\mathcal{A}$ with M and N proper $\mathcal{A} - E$-bimodules, then f is a morphism of $\mathcal{A} - E$-bimodules if and only if f^0 is a morphism of $R - E$-bimodules and for all $v \in V(\mathcal{A})$, $f^1(v) : M \to N$ is a morphism of right E-modules.

Theorem 6.3. *Assume $\mathbf{C^1_n}(\operatorname{proj}\Lambda)$ is not of wild representation type, then given a natural number d, there is a finite set of full and faithful functors $G_i : \text{Rep}\mathcal{B}_i \to \mathbf{C^1_n}(\operatorname{Proj}\Lambda)$, $i = 1, ..., t$, such that:*
i) the tbocses $\mathcal{B}_i = (R_i, W^i, \delta_i)$ are minimal triangular tbocses;
ii) for $i = 1, ..., t$ there are complexes $Y_i = (Y^j_i)$ with Y^j_i $\Lambda - R_i$ bimodules projectives on both sides and finitely generated over the right side with $F_i(N) \cong Y \otimes_{R_i} N$;
iii) for any $X \in \mathbf{C^1_n}(\operatorname{Proj}\Lambda)$ with $\operatorname{endol}(X) \leq d$ there is a $i \in \{1, ..., t\}$ and a $N \in \text{Rep}\mathcal{B}_i$ with $F_i(N) \cong X$.

Proof. We prove our claim by induction on n. First we consider the case $n = 1$. Clearly $\mathbf{C^1_1}(\operatorname{Proj}\Lambda) \cong \operatorname{Proj}\Lambda$.

Take the tbocs $\mathcal{U} = (\Lambda, 0, 0)$, then Rep$\mathcal{U} = \operatorname{Mod}\Lambda$. Consider $X =_\Lambda \Lambda$, here $\operatorname{End}_\Lambda(X)^{op} \cong S \oplus \operatorname{rad}\Lambda$. We have the tbocs $\mathcal{U}^X = (S, W, \delta)$, where $W_0 = 0$, $W_1 = (\operatorname{rad}\Lambda)^*$ and δ is the extension to $T_S(W)$, using Leibnitz rule, of the comultiplication $(\operatorname{rad}\Lambda)^* \to (\operatorname{rad}\Lambda)^* \otimes_S (\operatorname{rad}\Lambda)^*$. There is a full and faithful functor $F^X : \text{Rep}\mathcal{U}^X \to \operatorname{Mod}\Lambda$. For $M \in \text{Rep}\mathcal{U}^X$, $F^X(M) = \Lambda \otimes_S M$. The full and faithful functor F^X induces an equivalence $F^X : \text{Rep}\mathcal{U}^X \to \operatorname{Proj}\Lambda \cong \mathbf{C^1_1}(\operatorname{Proj}\Lambda)$. Here $\mathcal{U}^X$ is a minimal tbocs, thus we have i), $X = \Lambda$ is a $\Lambda - S$-bimodule projective fintely generated on both sides, thus we have ii), here $F^X : \text{Rep}\mathcal{U}^X \to \operatorname{Proj}\Lambda$ is an equivalence and then we have iii).

Assume now our result proved for n, we will prove it for $n + 1$.

By the induction hypothesis for $i = 1, ..., l$ there are full and faithful functors $F_i : \text{Rep}\mathcal{A}_i \to \mathbf{C^1_n}(\operatorname{Proj}\Lambda)$ with $\mathcal{A} = (R_i, W^i, \delta_i)$ minimal tbocses and complexes Y_i of $A(\mathcal{A}) - R_i$-bimodules projectives finitely generated over the right side such that $Y^j_i = 0$ for j outside the interval $[1, n]$ and $F_i(N) \cong Y_i \otimes_{R_i} N$. Moreover if $X \in \mathbf{C_n}(\operatorname{Proj}\Lambda)$ and $\operatorname{endol}(X) \leq d'$, there is a $N \in \text{Rep}\mathcal{A}_i$ for some $i \in [1, l]$ with $F_i(N) \cong X$.

The functors $F_i : \mathrm{Rep}\mathcal{A}_i \to \mathbf{C^1_n}(\mathrm{Proj}\,\Lambda)$ induce linear transformations $t_{F_i} : D(\mathcal{A}_i) \to \mathbb{Q}^{mn}$, such that for $N \in \mathrm{rep}\,\mathcal{A}_i$, $c(F_i(N)) = t_{F_i}(\underline{\dim} N)$.

Take P a projective indecomposable Λ-module and suppose $Z(P,i) \in \mathrm{Rep}\mathcal{A}$ is such that $F_i(Z(P,i)) \cong S(P)$. Then $t_{F_i}(\underline{\dim} Z(P,i)) = (\underline{\dim} P/\mathrm{rad}P; 0; ...; 0)$. Take $f_{i,j}$ the only primitive central idempotent of R_i such that $f_{i,j}Z(P,i) \neq 0$. Then if $R_i f_{i,j}$ is not k, there are infinitely many non-isomorphic indecomposable objects T_s in $\mathrm{Rep}\mathcal{A}_i$ such that $\underline{\dim} T_s = \underline{\dim} Z(P,i))$. But then applying F_i this implies that there are infinitely many non-isomorphic indecomposable objects $F_i(T_s)$ in $\mathrm{Rep}\mathcal{A}$ with $\underline{\dim} F_i(T_s) = (\underline{\dim} P; 0; ...; 0)$, which is not possible. Therefore $Rf_{i,j} = k$. Take now f_i the sum of all possible $f_{i,j}$ as before. Then $R_i f_i$ is a semisimple k-algebra.

Now for $i \in [1,t]$ take $\mathcal{L}_i$ the category of radical morphisms $u : Z_2 \to Z_1$ in $\mathrm{Rep}\mathcal{A}_i$ with $f_i Z_2 = Z_2$. By Theorem 5.2 there is an equivalence of k-categories $G_i : \mathrm{Rep}\mathcal{B}_i \to \mathcal{L}_i$, with $\mathcal{B}_i = (S_i, W_{\mathcal{B}_i}, \delta_{\mathcal{B}_i})$ a triangular tbocs. Since $\mathcal{A}$ is not of wild representation type then each $\mathcal{B}_i, i \in [1,t]$ is not of wild representation type. Then there are full and faithful functors $F_{i,j} : \mathrm{Rep}\mathcal{A}_{i,j} \to \mathrm{Rep}\mathcal{B}_i$ for $j \in [1, l(i)]$ with $\mathcal{A}_{i,j} = (S_{i,j}, W_{i,j}, \delta_{i,j})$ minimal triangular tbocses such that for all $M \in \mathrm{Rep}\mathcal{B}_i$ with $\mathrm{endol}(M) \leq d'$ there is a $N \in \mathrm{Rep}\mathcal{A}_{i,j}$ for some $j \in [1, l(j)]$ with $F_{i,j}(N) \cong M$.

The functor $F_i : \mathrm{Rep}\mathcal{A}_i \to \mathrm{Rep}\mathcal{A}$ induces a full and faithful functor $\hat{F}_i : \mathcal{L}_i \to \mathcal{M}_n$, $\hat{F}_i(u : Z_2 \to Z_1) = F_i(u) : F_i(Z_2) \to F_i(Z_1)$.

We have the following full and faithful functors:

$$\mathrm{Rep}\,\mathcal{B}_{i,j} \xrightarrow{F_{i,j}} \mathrm{Rep}\,\mathcal{B}_i \xrightarrow{G_i} \mathcal{L}_i \xrightarrow{\hat{F}_i} \mathcal{M}_n \xrightarrow{G} \mathbf{C^1_{n+1}}(\mathrm{Proj}\,\Lambda).$$

We have the proper $\mathcal{B}_{i,j} - R_{i,j}$-bimodule $F_{i,j}(R_{i,j}) = V_{i,j}$. Then $V_{i,j}$ is a $A(\mathcal{B}_{i,j}) - R_{i,j}$-bimodule. We recall that

$$A(\mathcal{B}_i) = \begin{pmatrix} R_i & W^i f_i \\ 0 & f_i R_i f_i \end{pmatrix},$$

$V_{i,j} = (V^1_{i,j}, V^2_{i,j}; h_{i,j})$ with $V^1_{i,j}$ and $V^2_{i,j}$ $R_i - R_{i,j}$-bimodules finitely generated projectives over the right side. The morphism $h_{i,j} : W^i f_i \otimes_{R_i} V^2_{i,j} \to V^1_{i,j}$ is a morphism of $R_i - R_{i,j}$-bimodules. Then $V^1_{i,j}$ and $V^2_{i,j}$ are proper $\mathcal{A}_i - R_{i,j}$-bimodules and $\phi_{i,j} = (0, \phi^1_{i,j}) : V^2_{i,j} \to V^1_{i,j}$ with $\phi^1_{i,j}(w)(x) = h_{i,j}(w)(m)$ for $w \in W^i_1, x \in V^2_{i,j}$. Since $\phi_{i,j}$ is a morphism of $R_i - R_{i,j}$-bimodules, $h_{i,j}$ is a morphism of $\mathcal{A}_i - R_{i,j}$-bimodules.

By definition $G_i(V_{i,j}) = h_{i,j} : V^2_{i,j} \to V^1_{i,j}$, $\hat{F}_i(G_i(V_{i,j})) = F_i(h_{i,j}) : Y_i \otimes_{R_i} V^2_{i,j} \to Y_i \otimes_{R_i} V^1_{i,j}$.

Now $f_i V^2_{i,j} = V^2_{i,j}$, then $(Y_i \otimes_{R_i} V^2_{i,j})^1 = Y^1_i \otimes_{R_i} V^2_{i,j}$ and $(Y_i \otimes_{R_{i,j}} V_{i,j})^s = 0$ for $s \neq 1$, $(Y_i \otimes_{R_i} V^1_{i,j})^s = Y^s_i \otimes_{R_i} V^1_{i,j}$ for $s \in \mathbb{Z}$, $F_i(h_{i,j})^1 = u_{i,j}$, $F_i(h_{i,j})^s = 0$ for $s \neq 1$.

For $Z = G\hat{F}_i G_i F_{i,j}(R_{i,j})$ we have $Z^s = 0$ for s outside the interval $[1, n+1]$, $Z^1 = Y^1_i \otimes_{R_i} V^2_{i,j}$, $Z^2 = Y^1_i \otimes_{R_i} V^1_{i,j}$,..., $Z^{n+1} = Y^n_i \otimes_{R_i} V^1_{i,j}$; and $d^1_Z = u_{i,j}$, $d^s_Z = d^{s-1}_{Y_i} \otimes 1$ for $s \in [2, n+1]$.

For $M \in \mathrm{Rep}\,\mathcal{B}_{i,j}$ we have $G\hat{F}_i G_i F_{i,j}(M) \cong Z \otimes_{R_{i,j}} M$.

We shall see that the functors $H_{i,j} = G\hat{F}_i G_i F_{i,j} : \mathrm{Rep}\,\mathcal{B}_{i,j} \to \mathbf{C^1_{n+1}}(\mathrm{Proj}\,\Lambda)$ satisfy the conditions i), ii) and iii). Here the tbocs $\mathcal{B}_{i,j}$ is triangular minimal, thus we have i). Now for Z we have that for $s \in [1, n+1]$, Z^s is a $\Lambda - R_{i,j}$-bimodule projective on both sides and finitely generated over the right side and for $M \in \mathrm{Rep}\,\mathcal{B}_{i,j}$, $H_{i,j}(M) \cong Z \otimes_{R_{i,j}} M$, thus we have ii).

For proving iii) take $X \in \mathbf{C^1_{n+1}}(\mathrm{Proj}\,\Lambda)$ with endol$(X) \leq d$. Then $X \cong G(X_2 \xrightarrow{u} X_1)$ with $X_2 = S(P), X_1 \in \mathbf{C^1_n}(\mathrm{Proj}\,\Lambda)$. Consider $E = \mathrm{End}_{\mathbf{C_n}(\mathrm{Proj}\,\Lambda)}(X)^{op}$, X_1 and X_2 are $\mathbf{C_n}(\mathrm{Proj}\,\Lambda) - E$-bimodules and endol$(X) = \mathrm{length}_E X_1 + \mathrm{length}_E X_2$. Then endol$(X_1) \leq \mathrm{length}_E X_1$ and endol$(X_2) \leq \mathrm{length}_E X_2$. Therefore endol$(X_1 \oplus X_2) \leq$ endol(X_1) + endol$(X_2) \leq d$. Then there is an i and $N_1, N_2 \in \mathrm{Rep}\,\mathcal{A}_i$ such that $F_i(N_1) \cong X_1, F_i(N_2) \cong X_2$. Since F_i is a full functor, there is a morphism $v = (0, v^1) : N_1 \to N_2$ such that $F_i(v)$ is isomorphic to u. The morphism v is an object of $\mathcal{L}_i$. Clearly v is an $\mathcal{L}_i - E$-bimodule with $\hat{F}_i(v) \cong u$. Since G_i is an equivalence there is a $N \in \mathcal{B}_i$ with $G_i(N) \cong v$. We may assume $N = (N_1, N_2; h)$, then endol$(N) \leq$ endol(N_1) + endol(N_2) = endol(X_1) + endol$(X_2) \leq d$. Then there is a j and an object $M \in \mathrm{Rep}\,\mathcal{B}_{i,j}$ with $F_{i,j}(M) \cong N$. Therefore $H_{i,j}(M) \cong X$, this proves iii). □

Proof of Theorem 1.1 Suppose $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ is not of wild representation type. Therefore $\mathbf{C^1_m}(\mathrm{proj}\,\Lambda)$ is not of wild representation type, consequently by Theorem 6.3, given a non negative integer d, there is a finite set of full and faithful functors $G_i : \mathrm{Rep}\,\mathcal{B}_i \to \mathbf{C^1_n}(\mathrm{Proj}\,\Lambda)$, $i = 1, ..., t$ with conditions i), ii) and iii). Using the notation of Theorem 6.3, for $i \in \{1, ..., t\}$ we consider T_i the set of central primitive idempotents $f_{i,j}$ in R_i with $f_{i,j}R_i \neq kf_{i,j}$. For each $f_{i,j} \in T_i$ we have $Yf_{i,j} \in \mathbf{C^1_n}(\mathrm{Proj}\,\Lambda)$. Each $Y^u f_{i,j}$ is a $\Lambda - R_i f_{i,j}$ bimodule projective finitely generated as right $R_i f_{i,j}$-module, since $R_i f_{i,j}$ is a rational k-algebra, then $Y^u f_{i,j}$ is a free finitely generated right $R_i f_{i,j}$-module. Then for almost all isomorphism classes $[X]$ of indecomposable objects in $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ with $\dim_k X \leq d$, we may assume $X \in \mathbf{C^1_m}(\mathrm{proj}\,\Lambda)$ and endol$(X) = \dim_k X \leq d$. Therefore for almost all such $[X]$ we have $X \cong Y_i \otimes_{R_i f_{i,j}} S(\lambda)$ for some $\lambda \in k$ and $f_{i,j} \in T_i$. This proves that $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ is of tame representation type. □

The following result implies Theorem 1.2.

Theorem 6.4. *Assume that* $\mathbf{C^1_m}(\mathrm{proj}\,\Lambda)$ *is not of wild representation type. Then given a natural number d for almost all indecomposable object* $X \in \mathbf{C^1_m}(\mathrm{proj}\,\Lambda)$ *with* $\dim_k X \leq d$ *there is an* $\mathcal{E}$*-almost split sequence:*

$$X \to E \to X.$$

Proof. We may assume X is not $\mathcal{E}$-projective then by Theorem 8.5 of [2], there is an $\mathcal{E}$-almost split sequence:

$$A(X) \to E \to X$$

in $\mathbf{C^1_m}(\mathrm{proj}\,\Lambda)$.

We will prove first that there is a constant $c(\Lambda)$ depending only on the algebra Λ such that for any $Y \in \mathbf{C^1_m}(\mathrm{proj}\,\Lambda)$, $\dim_k A(Y) \leq c(\Lambda)\dim_k Y$. Take $L = \dim_k \Lambda$, and the Nakayama functor $\nu : \mathrm{proj}\,\Lambda \to \mathrm{inj}\,\Lambda$. We recall that if $1 = \sum_{i=1}^n e_i$ is a decomposition of the identity of Λ into orthogonal primitive idempotents then $\nu(\Lambda e_i) = D(e_i\Lambda)$. Therefore if $P = \oplus_i n_i \Lambda e_i$, then $\nu(P) = \oplus_i n_i D(e_i\Lambda)$. Thus $\dim_k \nu(P) = \sum_i n_i \dim_k D(e_i\Lambda) \leq \sum_i n_i L \leq L(\sum_i n_i \dim_k \Lambda e_i) = L\dim_k P$. If $W = (W^i, d^i_W)$ is a complex of finitely generated projective Λ- modules then $\nu(W) = (\nu(W^i), \nu(d^i_W))$. If in addition W is a finite complex $\dim_k \nu(W) = \sum_i \dim_k \nu(W^i) \leq L\dim_k W$.

Now choose a quasi-isomorphism $q : Z \to \tau^{\leq m}(\nu(X)[-1])$, with $Z = (Z^i, d_Z)$ such that $\mathrm{Im} d^i_Z \subset \mathrm{rad} Z^{i+1}$.

We have $\dim_k H^j(Z) = \dim_k H^j(\tau^{\leq m} X[-1]) \leq L\dim_k X$. Now $A(X) \cong F(Z)$ in $\mathbf{C^1_m}(\mathrm{proj}\,\Lambda)$, thus $\dim_k A(X) \leq c(\Lambda)\dim_k X$ with $c(\Lambda) = L(mL + (m-1)L^2 + \ldots 2L^{m-1} + L^m)$. This proves our claim.

Given a natural number d, we take $d' = 2(1 + c(\Lambda))d$. By Theorem 6.3 there is a finite number of full and faithful functors $F_i : \mathrm{Rep}\mathcal{B}_i \to \mathbf{C^1_m}(\mathrm{Proj}\,\Lambda)$ with $\mathcal{B}_i = (R_i, W^i, \delta_i)$ minimal triangular tbocses such that for any $Y \in \mathbf{C^1_m}(\mathrm{Proj}\,\Lambda)$ with $\mathrm{endol}Y \leq d'$ there is a $W \in \mathrm{Rep}\,\mathcal{B}_i$ with $F_i(W) \cong Y$. Consider now the family $\mathcal{S}$ of objects in $\mathbf{C^1_m}(\mathrm{proj}\,\Lambda)$ which are isomorphic to some $F_i(f_s R_i)$ with f_s central primitive idempotent of R_i such that $f_s R_i = k$. In the above family there is only a finite number of isomorphism classes.

Take now an indecomposable object $X \in \mathbf{C^1_m}(\mathrm{proj}\,\Lambda)$ which is not in $\mathcal{S}$ with $\dim_k X \leq d$. Suppose moreover that X is not $\mathcal{E}$-projective. Then there is an $\mathcal{E}$-almost split sequence:

$$a \quad Y \to E \to X,$$

here $\mathrm{endol}(X \oplus E \oplus Y) \leq \dim_k(X \oplus E \oplus Y) \leq d'$, then there is a $U \in \mathrm{Rep}\,\mathcal{B}_i$ with $F_i(U) \cong (X \oplus E \oplus Y)$. Therefore there are objects N, M, W in $\mathrm{Rep}\,\mathcal{B}_i$ with $F_i(M) \cong X, F_i(N) \cong Y, F_i(W) \cong E$. Since F_i is full and faithful, thus there is an almost split sequence $N \to W \to M$ whose image is isomorphic to a. Here M is not isomorphic to some $f_s R_i$ with f_s central primitive idempotent of R_i such that $f_s R_i = k$ thus $N \cong M$ which implies that $X \cong Y$. □

7. Generic Complexes

Here we consider generic complexes in the sense of section 5 of [16]. For Λ a derived tame algebra we shall see the relations between one-parameter families of objects in $\mathcal{D}^b(\Lambda)$ and generic complexes in $\mathcal{D}^b(\mathrm{Mod}\,\Lambda)$.

Definition 7.1. *A complex $X \in \mathcal{D}^b(\mathrm{Mod}\,\Lambda)$ is called endofinite if $H^i(X)$ has finite length as $E(X) = \mathrm{End}_{\mathcal{D}^b(\mathrm{Mod}\,\Lambda)}(X)$-module for all $i \in \mathbb{Z}$.*

An endofinite complex X is called generic if it is indecomposable and it is not isomorphic to a bounded complex of finitely presented Λ-modules.

The homology endolength of an endofinite X object of $\mathcal{D}^b(\mathrm{Mod}\,\Lambda)$ is defined as:

$$\mathbf{h}\mathrm{endol}X = (\mathrm{length}_{E(X)} H^i(X))_{i \in \mathbb{Z}}.$$

Definition 7.2. *An infinite family $\mathcal{F}$ of pairwise non-isomorphic indecomposable objects in $\mathcal{D}^b(\Lambda)$, (respectively in $\mathbf{C_n}(\mathrm{mod}\,\Lambda)$) is called one-parameter family if there is a rational k-algebra R and a bounded complex X of $\Lambda - R$-bimodules (respectively X a $\mathbf{C_n}(\mathrm{Proj}\,\Lambda) - R$-bimodule) with each X^i is free finitely generated over R, such for any $M \in \mathcal{F}$, there is a $\lambda \in S(R)$ with $M \cong X \otimes_R k[x]/(x - \lambda)$. We say that $\mathcal{F}$ is parametrized by Y.*

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two one-parameter families of complexes in $\mathbf{C_n}(\mathrm{mod}\,\Lambda)$ the set $\mathcal{F}_{1,2}$ of those $X \in \mathcal{F}_1$ such that there is a $Y \in \mathcal{F}_2$ with $X \cong Y$ is either finite or cofinite in $\mathcal{F}_1$. The relation between the one-parameter families defined by $\mathcal{F}_1 \approx \mathcal{F}_2$ if the set $\mathcal{F}_{1,2}$ is infinite is an equivalence relation. We say that $\mathcal{F}_1$ is equivalent to $\mathcal{F}_2$ if $\mathcal{F}_{1,2}$ is infinite.

Definition 7.3. *If X is a bonded complex of $\Lambda - k(x)$-bimodules a realization of X is a bounded complex Y of $\Lambda - R$-bimodules, with R a rational k-algebra such that $X \cong Y \otimes_R k(x)$ in the category $\mathcal{D}^b(\mathrm{Mod}\,\Lambda)$.*

Theorem 7.4. *Let Λ be a derived tame k-algebra, with k algebraically closed field, suppose X is a generic complex in $\mathcal{D}^b(\mathrm{Mod}\,\Lambda)$. Then:*
i) X is isomorphic to P a bounded complex of finitely generated $\Lambda - k(x)$-bimodules, moreover $\mathbf{h}\mathrm{endol}X = (\dim_{k(x)} H^i(P))$;
ii) there is a rational k-algebra R and a complex Y of $\Lambda - R$-bimodules free finitely generated over the rigth side such that $Y \otimes_R k(x) \cong X$ in $\mathcal{D}^b(\mathrm{Mod}\,\Lambda)$ and $Y \otimes_R - : \mathrm{mod}\,R \to \mathcal{D}^b(\mathrm{mod}\,\Lambda)$ preserves indecomposables and isomorphism classes.
Moreover, if $\mathcal{F}$ is a one-parameter family of indecomposable objects in $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$, then there is a generic complex $X \in \mathcal{D}^b(\mathrm{Mod}\,\Lambda)$ and a realization Y of X such that $\mathcal{F}$ is equivalent to a one-parameter family parametrized by $Y \otimes_R R/(p)^n$ with p a prime element in R.

Proof. We may assume that for $(h_i) = \mathbf{h}\mathrm{endol}X^\bullet$ we have $h_i = 0$ for $i \leq 2$ and $i > m$, $h_2 \neq 0$. Take now $P \in \mathbf{K}^{\leq \mathbf{m},\mathbf{b}}(\mathrm{Proj}\,\Lambda)$ quasi-isomorphic to X. Then $H^i(P) = 0$ for $i \leq 2$. We have $F(P)$ is indecomposable in $\mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\mathrm{Proj}\,\Lambda)$, with F the functor given after Lemma 2.2. Now $F(P) = Q = (Q^i, d^i_Q)$ is a complex such that each Q^i has finite lengh as $\mathrm{End}_Q(Q)$-module, then Q has endofinite length d. Since we have an equivalence $F : \mathcal{L}_m \to \overline{\mathbf{C}_{\mathbf{m}}}(\mathrm{Mod}\,\Lambda)$, Q is a generic object. By Theorem 6.3 there is a full and faithful functor $G : \mathrm{Rep}\,\mathcal{B} \to \mathbf{C}^{\mathbf{1}}_{\mathbf{n}}(\mathrm{Proj}\,\Lambda)$ with $\mathcal{B} = (S, W, \delta)$ a minimal triangular tbocs and $G(M) \cong Q$ for some $M \in \mathrm{Rep}\,\mathcal{B}$. Thus M is a generic object in $\mathrm{Rep}\,\mathcal{B}$, then there is a central primitive idempotent $f \in S$ such that $M = k(x)f$.

By ii) of Theorem 6.3 there is a complex Z of $\Lambda - S$-bimodules projectives on both sides and finitely generated over the right side such that for all $N \in \mathrm{Rep}\,\mathcal{B}$, $F(N) \cong Z \otimes_S N$, thus $Q \cong Z \otimes_S fk(x) \cong Zf \otimes_{fSf} k(x)$. Here $R = fSf$ is a rational k-algebra and $Y = Zf$ is complex of projective right R-module then Y is a complex of free finitely generated right R-modules. Our complex Y satisfies the hypothesis of Corollary 2.7, therefore since $Q \cong Y \otimes_R k(x)$, the morphism $d^1_Q : Q^1 \to Q^2$ is a monomorphism. But $d^1_P : P^1 \to P^2 = d^1_Q : Q^1 \to Q^2$, then d^1_P is a monomorphism. But $H^1(P) = 0$, then $d^0_P = 0$, but this implies that $P^j = 0$ for $j \leq 0$, consequently $P = Q$. We have that the radical of $\mathrm{End}_{\mathcal{B}}(M)$ is nilpotent and $\mathrm{End}_{\mathcal{B}}(M)/\mathrm{rad}\mathrm{End}_{\mathcal{B}}(M) \cong k(x)$, thus for $E_P = \mathrm{End}_{\mathrm{C_m(Proj}\,\Lambda)}(P)$ we have $E_P/\mathrm{rad}E_P \cong k(x)$. From this we obtain i). Since G is a full and faithful functor, we obtain ii).

For the last statement of our theorem suppose that $\mathcal{F}$ is a one-parameter family in $\mathcal{D}^b(\Lambda)$. We may assume that there is a fixed $\mathbf{h} = (h_i)$ such that for all $X \in \mathcal{F}$, $\mathbf{h}\mathrm{dim}X = \mathbf{h}$. By Theorem 2.4 we may assume that all $X \in \mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\mathrm{proj}\,\Lambda)$ and there is a fixed d such that $\mathrm{endol}X \leq d$. By Theorem 6.3 there are full and faithful functors $G_i : \mathrm{Rep}\,\mathcal{B}_i \to \mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\mathrm{proj}\,\Lambda)$ with $\mathcal{B}_i = (R_i, W_i, \delta_i)$ minimal tbocses such that for all $X \in \mathcal{F}$ there is a $N \in \mathrm{Rep}\,\mathcal{B}_i$ with $F_i(N) \cong X$. Moreover there are complexes Y_i such that for $M \in \mathrm{Rep}\,\mathcal{B}_i$, $G_i(M) \cong Y_i \otimes_{R_i} M$. In $\mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\mathrm{proj}\,\Lambda)$ there are one-parameter families parametrized by the complexes $Y_i f_{i,j} R_i/(p)^n$ with p prime element of $R_i f_{i,j}$ and $f_{i,j}$ central primitive idempotents of R_i with $R_i f_{i,j} \neq k f_{i,j}$. Almost all objects in $\mathcal{F}$ are in one of these one-parameter families, then $\mathcal{F}$ is equivalent with one of these families. This proves our result. ☐

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